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MATRIX NORMS

By G. M. PETERSEN (*Albuquerque, New Mexico*)

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1. In this paper we investigate the regular summation matrices

$$A = (a_{mn})$$

that have the additional property

$$\lim_{m\to\infty} \max_n |a_{mn}| = 0.$$

I shall call the class of these matrices $\mathfrak{A}$. In this first section I consider sequences of such matrices $\{A^k\}$ such that A^k sums all bounded sequences that are summable A^{k-1} and show that there is a sequence of regular iterations $\{B^k\}$, the matrix B^k being equivalent to A^k for bounded sequences. In the second section, I prove some properties of the norm of the matrix. Matrices of class $\mathfrak{A}$ have been investigated extensively by Lorentz (**2**).

A matrix $B = (b_{mn})$ is said to be 'b-stronger than $A = (a_{mn})$' if every bounded sequence that is A-summable is also B-summable. If B is b-stronger than A and A is b-stronger than B, then A and B are 'b-equivalent'. The matrix B is a-stronger than A if all sequences (bounded and unbounded) that are A-summable are also B-summable; if A is also a-stronger than B, the two matrices are 'a-equivalent'. Two matrices are 'b-consistent' if every bounded sequence summed by them both is summed to the same sum. The following theorem has been proved by Brudno (**1**); see also (**5**).

Theorem 1. *If $B = (b_{mn})$ is regular and b-stronger than $A = (a_{mn})$, then B must be b-consistent with A.*

We first prove

Lemma 1. *For every regular matrix $A = (a_{mn})$ and every strictly increasing integral-valued function $k(m)$, there is a matrix $A' = (a'_{mn})$ that is b-equivalent to A and such that $a'_{m,k(m)} \neq 0$, $a'_{mn} = 0$ $(n > k(m))$.*

Proof. C. Goffman and the author have shown (**3**) that, if $A = (a_{mn})$ is infinite-rowed, there is a finite-rowed matrix b-equivalent to A, and hence we can assume that A is finite-rowed. We may also assume that $a_{1,1} \neq 0$, $a_{1,n} = 0$ $(n > 1)$. We can rearrange the rows so that the number of the last non-zero element $\lambda(m)$ is a strictly increasing function of m. We now form a new matrix $A'' = (a''_{mn})$ as follows: we

first repeat the first row of A (i.e. $a_{1,n}$) ν_1 times where $k(\nu_1) \geqslant \lambda(2)$, we then repeat the second row ν_2 times, where $k(\nu_1+\nu_2) \geqslant \lambda(3)$, and then introduce the third row of A and so on. In forming $A' = (a'_{m,n})$, we have $a'_{mn} = a''_{mn}$ for all n if $a''_{m,k(m)} \neq 0$. If $a''_{m,k(m)} = 0$, $a'_{mn} = a''_{mn}$ for $n \neq k(m)$, $a'_{m,k(m)} = 1/m$. It is evident that A'' and A are b-equivalent, but so are A' and A'' since, if $|s_n| < H$,

$$|\sum (a'_{mn}-a''_{mn})s_n| < \frac{H}{m}.$$

This completes the proof of the lemma.

The value of the *counting function* $\omega(n)$ of a sequence $\{n_k\}$ of integers is, for a given n, the number of n_k satisfying the inequality $n_k \leqslant n$. Let $\Omega(n)$ be a fixed positive function increasing to $+\infty$ with n. The function $\Omega(n)$ is called a *summability function* of $A = (a_{mn})$ if

$$\lim_{m\to\infty} \sum |a_{mn}| = 0 \quad (n \in n_k)$$

for any sequence $\{n_k\}$ for which $\omega(n) \leqslant \Omega(n)$. For instance, each function $\Omega(n) = o(n)$ is a summability function of the method $(C, 1)$. Lorentz (**2**) showed that

THEOREM 2. *A regular method $A = (a_{mn})$ belongs to $\mathfrak{A}$ if and only if it has a summability function $\Omega(n)$.*

We have immediately the lemma:

LEMMA 2. *If $A = (a_{mn})$ is of the class $\mathfrak{A}$ and $b_{mn} = a_{mn}$ $(n \notin n_k)$, $b_{m,n_k} = 0$ $(k = 1, 2,...)$, then A and $B = (b_{mn})$ are b-equivalent, the counting function of $\{n_k\}$ being dominated by $\Omega(n)$.*

We now prove

LEMMA 3. *Let $\{A^r\}$ $(r = 1, 2,...)$ be a sequence of matrices of type $\mathfrak{A}$, $A^r = (a^r_{mn})$, such that A^r is b-stronger than A^{r-1}. Let $\sum |a^r_{mn}| \leqslant M$ for every m and r. There exists a sequence of matrices $\{B^r\}$ such that B^r is b-equivalent to A^r, and B^r is a-stronger than B^{r-1}. Also*

$$b^r_{m,\lambda_r(m)} \neq 0, \qquad b_{mn} = 0 \quad (n > \lambda_r(m))$$

and

$$\text{either} \quad \lambda_r(m+1) = \lambda_r(m)+1 \quad \text{or} \quad \lambda_r(m+1) = \lambda_r(m)+(r+1);$$

in the latter case $b^r_{i,\lambda_r(m)+1} = ... = b_{i,\lambda_r(m)+r} = 0$ $(i = 1, 2,...)$.

Proof. It is clear from Lemma 1 that we may assume $A^r = (a^r_{mn})$ to be finite-rowed and, if $a_{m,k_r(m)} \neq 0$, $a_{mn} = 0$ $(n > k_r(m))$, then

$$k_r(m) = \nu_m + r,$$

where $\omega(\nu)$, the counting function of $\{\nu_m\}$, satisfies $\omega(\nu) \leqslant \Omega(\nu)$, and $\Omega(\nu)$ is a summability function of A^1. A summability function of A^1 is a summability function of A^r $(r = 1, 2)$. We shall further assume that

$$\nu_{m+1}-\nu_m \geqslant \nu_m-\nu_{m-1}+1.$$

We construct a matrix $C^r = (c^r_{mn})$ such that

$$c^r_{mn} = a^r_{mn} \quad \text{when } n \neq k_r(i)-(r+1),\dots, k_r(i)-1, \text{ for all } i,$$

$$c^r_{mn} = 0 \quad \text{when } n = k_r(i)-(r+1),\dots, k_r(i)-1, \text{ for all } i.$$

By Lemma 2 the matrix C^r is b-equivalent to A^r. Let $C^r_\mu = \sum c^r_{\mu n} s_n$. Then we construct

$$B^r = (b^r_{mn}), \qquad B^r_\mu = \sum b^r_{\mu n} s_n$$

as follows. Take m' so that $\nu_{m'+1}-\nu_{m'} > r$; then, for $m \geqslant m'$, $B^r_\mu = C^r_m$ if $\mu = \nu_m-(m-m')r$; if $\nu_m-(m-m')r < \mu < \nu_{m+1}-(m+1-m')r$, then $B^r_\mu = \left(1-\frac{1}{i+1}\right)C^r_m+\frac{1}{i+1}C^{r+i}_m$, where $i = \mu-\nu_m+(m-m')r$; for $\mu < m'$, let $B^r_\mu = s_\mu$.

It is clear that B^r is regular. Moreover, if C^r does not sum a sequence, $\{B^r_{\nu_m-(m-m')r}\}$ diverges, so that, if it is B^r-summable, a sequence must also be C^r-summable. If a bounded sequence is C^r-summable to s, it is also summable C^{r+i} to s $(i \geqslant 1)$, by Theorem 1. Suppose that $|s'_n| \leqslant H$ for all n and $\sum |c^r_{mn}| \leqslant M'$ for all m and r. Then choose j such that $HM'/j+1 \leqslant \epsilon$; for $0 \leqslant i \leqslant j$ there is an m'' such that $\max_{0\leqslant i\leqslant j} |C^{r+i}_m-s| \leqslant \epsilon$ for $m \geqslant m''$. Thus, if

$$\nu_m-(m-m')r \leqslant \mu \leqslant \nu_m-(m-m')r+j \qquad (m \geqslant m''),$$

then
$$|B^r_\mu-s| \leqslant \left(1-\frac{1}{i+1}\right)\epsilon+\frac{1}{i+1}\epsilon = \epsilon.$$

But, if

$$\nu_m-(m-m')r+j \leqslant \mu < \nu_{m+1}-(m+1-m')r \qquad (m \geqslant m''),$$

then
$$|B^r_\mu-s| \leqslant \left(1-\frac{1}{i+1}\right)\epsilon+\frac{1}{i+1}M'H$$

$$\leqslant \left(2-\frac{1}{i+1}\right)\epsilon \leqslant 2\epsilon.$$

Hence $\{s'_n\}$ is B^r-summable, and B^r and C^r are b-equivalent. An unbounded sequence will not be B^r-summable if it is not C^{r+i}-summable for every $i \geqslant 0$ to the same sum. In fact, there must be an $\{\epsilon_m\}$ such

that $(i+1)^{-1}|C_m^{r+i}-s| \leqslant \epsilon_m$ for all $0 \leqslant i \leqslant \nu_{m+1}-\nu_m-r$ and $\lim_{m\to\infty} \epsilon_m = 0$. For, if there were a sequence $\{i(m)\}$ such that

$$\frac{1}{i+1}|C_m^{r+i}-s| > \epsilon,$$

then clearly $\{B_\mu^r\}$ would diverge, where $\mu = \nu_m-(m-m')r+i(m)$. However, if $\{\epsilon_m\}$ exists, for

$$\nu_m-(m-m')r \leqslant \mu < \nu_{m+1}-(m+1-m')r,$$

then
$$|B_\mu^r-s| \leqslant \left(1-\frac{1}{i+1}\right)\epsilon_m+\epsilon_m \leqslant 2\epsilon_m,$$

and $\{B_\mu^r\}$ converges. On the other hand, if $\{s_n\}$ is an unbounded sequence that is B^r-summable, so that

$$\frac{1}{1+i}|C_m^{r+i}-s| \leqslant \epsilon_m$$

for all $0 \leqslant i < \nu_{m+1}-\nu_m-r$, then it must follow that

$$\frac{1}{i}|C_m^{r+1+(i-1)}-s| \leqslant \frac{1+i}{i}\epsilon_m \leqslant 2\epsilon_m$$

when $0 \leqslant i-1 < \nu_{m+1}-\nu_m-(r+1)$.

This means that, for a suitable m''',

$$\nu_m-(m-m''')(r+1) \leqslant \mu < \nu_{m+1}-(m+1-m''')(r+1) \quad (m > m'''),$$

$$|B_\mu^{r+1}-s| \leqslant \left(1-\frac{1}{1+i}\right)2\epsilon_m+2\epsilon_m,$$

and $\{s_n\}$ is B^{r+1}-summable. Hence B^{r+1} is a-stronger than B^r and b-equivalent to A^{r+1}.

We also have $k_{r+1}(m) = k_r(m)+1$ so that, if $b_{m,\lambda(m)}^r \neq 0$,

$$b_{mn}^r = 0 \quad (n > \lambda(m)), \qquad \lambda(\mu+1) = \lambda(\mu)+1$$

whenever $\nu_m-(m-m')r \leqslant \mu \leqslant \nu_{m+1}-(m+1-m')r-1$

or when $\mu < m'$. Also

$$\begin{aligned}\lambda(\nu_{m+1}-(m+1-m')r-1) &= k_{(\nu_{m+1}-\nu_m-r-1)+r}(m)\\ &= \nu_m+[\nu_{m+1}-\nu_m-r-1+r] = \nu_{m+1}-1\end{aligned}$$

and
$$\lambda(\nu_{m+1}-(m+1-m')r) = \nu_{m+1}+r,$$

so that

$$\lambda(\nu_{m+1}-(m+1-m')r)-\lambda(\nu_{m+1}-(m+1-m')r-1) = r+1.$$

However,

$$c_{in}^r = 0 \quad \text{when } k_r(m)-(r+1) \leqslant n \leqslant k_r(m)-1,$$

i.e.
$$c_{in}^r = 0 \quad \text{if } \nu_{m+1}-1 \leqslant n \leqslant \nu_{m+1}+r-1 \quad \text{for all } i.$$

Hence, if $\lambda(\mu+1)-\lambda(\mu) = r+1$, then

$$\mu = \nu_{m+1}-(m+1-m')r-1, \qquad \lambda(\mu) = \nu_{m+1}-1,$$

and
$$b_{\mu n} = \left(1-\frac{1}{1+i}\right)c^r_{\mu n}+\frac{1}{1+i}c^{r+i}_{\mu n} = 0$$

for all μ and $\nu_m \leqslant n \leqslant \nu_m+(r-1)$, i.e. for $\lambda_r(\mu)+1 \leqslant n \leqslant \lambda_r(\mu)+r$. This completes the proof of the lemma.

The iteration of a matrix $B = (b_{mn})$ with a matrix $A = (a_{mn})$ is defined by $t_k = \sum b_{km}\tau_m$ where $\tau_m = \sum a_{mn}s_n$. In short, the iteration of B with A or $B.A$ consists in applying the matrix (b_{mn}) to the sequence $\{\tau_m\}$ of A transforms of $\{s_n\}$.

We now prove

LEMMA 4. *There exists a sequence of regular matrices $\{D^r\}$ such that $B^r = D^r.B^{r-1}$ for every r.*

Proof. To find the matrix $D^r = (d^r_{mn})$ we must solve the equations

$$b^r_{mn} = \sum_{i=1}^{\lambda(m)} d^r_{mi}b^{r-1}_{in}.$$

We have $b^{r-1}_{mn} = 0$ for all m and $n = \nu_1+1,\ \nu_1+2,\ldots,\ \nu_1+(r-2),\ldots,$ $\nu_k+1,\ldots,\ \nu_k+(r-2)$, where ν_k is the last member of the sequence $\{\nu_m\}$ satisfying $\nu_k \leqslant \lambda(m)$; however, $b^r_{mn} = 0$ for these values of n also. Hence we need not consider these columns, and the coefficients of the remaining d^r_{mi} form a non-zero triangular determinant since

$$\lambda_{r-1}(m+1) = \lambda_{r-1}(m)+1 \quad (m \neq \nu_k)$$

or, if $m+1 = \nu_k+1$,

$$\lambda_{r-1}(m+1) = \lambda_{r-1}(m)+(r-1).$$

This means that we can solve for the corresponding d^r_{mi}. The matrix (d^r_{mi}) is regular since $\{B^r_m\}$ converges whenever $\{B^{r-1}_m\}$ converges. This completes our proof.

We collect the preceding lemmas in a theorem.

THEOREM 3. *Let $\{A^r\}$ $(r = 1,2,\ldots)$ be a sequence of matrices of type $\mathfrak{A}$, $A^r = (a^r_{mn})$, such that A^r is b-stronger than A^{r-1}. Let $\sum |a^r_{mn}| \leqslant M$ for every m and r. There exists a sequence of matrices $\{B^r\}$, $\sum |b^r_{mn}| \leqslant M$, such that B^r is b-equivalent to A^r and $B^r = D^r.B^{r-1}$ where $D^r = (d^r_{mn})$ is a regular matrix.*

2. The norm $h(A)$ of a matrix has been defined by Brudno (**1**) as

$$h(A) = \sup \sum |a_{mn}|.$$

The method $\mathscr{A}$ has a norm $\|\mathscr{A}\|$ given by $\|\mathscr{A}\| = \inf h(A)$, where the 'inf' is taken over all the matrix methods equivalent to $\mathscr{A}$ for bounded

sequences. The norm has the property that, if every $\mathscr{B}$-summable bounded sequence is also $\mathscr{A}$-summable, i.e. $\mathscr{A}$ is b-stronger than $\mathscr{B}$, then $\|\mathscr{A}\| \geqslant \|\mathscr{B}\|$. Brudno constructed a sequence of methods $\{\mathscr{A}_k\}$ such that $\mathscr{A}_{k+1}$ is b-stronger than $\mathscr{A}_k$ for every k and such that

$$\lim_{k\to\infty}\|\mathscr{A}_k\| = \infty.$$

The matrix A_1 is defined by the transformation $t_n = 2s_{2n}-s_{2n+1}$, and A_k by k iterations of the method. For each k there is a sequence of 1's and (-1)'s that A_k sums to 3^k, so that any matrix $E = (e_{mn})$ that is b-stronger than A_k must have

$$\liminf \sum |e_{mn}| \geqslant 3^k.$$

Since this would be true for all k, no regular method can be b-stronger than this sequence of matrices [see also (**6**)]. To show that such a sequence of 1's and (-1)'s exists, we observe that in each column of A_1 there is at most one non-zero element. Hence the same will be true of each matrix A_k. If $\lim \sum |a_{mn}^k|$ exists, therefore a sequence of 1's and (-1)'s exists that is summed to $\lim \sum |a_{mn}^k|$. However, for all m,

$$\sum |a_{mn}^2| = 2(2+1)+1(2+1) = (2+1)^2 = 3^2.$$

Likewise
$$\sum |a_{mn}^k| = (2+1)^k = 3^k.$$

I now wish to modify the definition of the norm of a matrix to

$$h'(A) = \limsup_{m\to\infty} \sum |a_{mn}|.$$

LEMMA 5. *The norm of $\mathscr{A}$ is defined equivalently by* $\|\mathscr{A}\| = \inf h(A)$ *or* $\|\mathscr{A}\| = \inf h'(A)$.

Proof. In the first place it is clear that for any matrix $A = (a_{mn})$ we can select those rows $\{m_\nu\}$ such that

$$\sum |a_{m_\nu n}| = K_\nu \geqslant U = h'(A),$$

and define a new matrix $B = (b_{mn})$ by

$$b_{mn} = a_{mn} \quad (m \neq m_\nu), \qquad b_{m_\nu n} = \frac{U}{K_\nu} a_{m_\nu n}.$$

However, A and B are b-equivalent: for let

$$t_m = \sum a_{mn} s_n, \qquad t'_m = \sum b_{mn} s_n, \qquad |s_n| \leqslant H.$$

Then
$$t_m - t'_m = 0 \quad (m \neq m_\nu), \qquad |t_{m_\nu} - t'_{m_\nu}| \leqslant H\left(1-\frac{U}{K_\nu}\right),$$

i.e. $\lim_{m\to\infty}|t_m - t'_m| = 0$. Hence $\inf h(A) = \inf h'(A)$ and our statement is proved.

THEOREM 4. *There is a sequence of matrices $\{B_k\}$ such that $h'(B_k) = 1$ for every k, B_{k+1} is a-stronger than B_k, and no regular matrix is a-stronger than every B_k.*

Proof. Consider the successive iterations of the matrices $A_k = (a_{mn}^k)$ of Brudno's example with the Cesàro matrix $C = (c_{mn})$, where

$$c_{mn} = m^{-1} \quad (n \leqslant m), \qquad c_{mn} = 0 \quad (n > m).$$

Denote the matrix $A_k . C$ by $B_k = (b_{mn}^k)$; then B_{k+1} is a-stronger than B_k. We have

$$b_{mn}^k = \sum_{\lambda_m}^{\lambda_{m+1}} a_{mi}^k c_{in} \quad (\lambda_r \equiv 2^{kr}),$$

the terms of the mth row of A_k being zero outside the limits of summation. Now consider the sequence $\{s_i\}$,

$$s_k = 2^k m c_{in} \quad (2^k m \leqslant i < 2^k(m+1)),$$

n remaining fixed. Since

$$s_i = \frac{2^k m}{2^k m+i},$$

we have $\lim_{i\to\infty} s_i = 1$ for fixed k. But A_k is regular, and this means that

$$\lim_{m\to\infty} \sum_{\lambda_m}^{\lambda_{m+1}} a_{mi}^k s_i = 1.$$

Hence $b_{m,n}^k > 0$ for large m. Since $c_{in} = i^{-1}$ for all n, this holds uniformly and, for some m, $b_{mn}^k > 0$ for all n unless some c_{in} in the sum are zero, i.e. for $n \leqslant m-2^k$. In these cases we have

$$|b_{mn}^k| \leqslant 1/2^k m \sum_{\lambda_m}^{\lambda_{m+1}} |a_{mi}^k| = \frac{3^k}{2^k m}.$$

Thus the sum of the last 2^k terms is $O(m^{-1})$, and the matrix is essentially a positive matrix, so that $h'(B_k) = 1$ for all k. Suppose that a matrix B is a-stronger than every B_k; since C is a triangular matrix, we can find a matrix $D = (d_{mn})$ such that $B = D.C$, and D would be a-stronger and hence b-stronger than A_k for every k. However, this is a contradiction, and so the theorem is proved.

For bounded sequences, if $t_m = \sum c_{mn} s_n$, then $|t_m - t_{m+k}| \to 0$ for every fixed k. However, the A_k methods do not sum bounded sequences with this property, and so we have that the matrices described in the theorem are all b-equivalent to C. The matrix C, therefore, is b-stronger than this set of matrices.

I want to express my thanks to Dr. M. A. Tropper of Queen Mary College, London, who kindly lent me her translation of Brudno's paper.

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MATRIX REPRESENTATIONS OF SEMIGROUPS

By G. B. PRESTON (*New Orleans*)

[Received 4 June 1957]

LET S be a semigroup with an identity element and let

$$\lambda = \{(a,b)\colon a,b \in S;\ Sa \subset Sb\},$$

$$\rho = \{(a,b)\colon a,b \in S;\ aS \subset bS\}.$$

Let $\mathscr{L} = \lambda \cap \lambda^{-1}$ and $\mathscr{R} = \rho \cap \rho^{-1}$. Then, in the notation of D. D. Miller and A. H. Clifford (**1**), as shown by J. A. Green (**2**),

$$\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L} = \mathscr{D}.$$

In a recent paper (**3**) M. P. Schützenberger introduced the concept of a $\mathscr{D}$-class of finite type: the $\mathscr{D}$-class D is of *finite type*† if

$$\mathscr{R} \cap \lambda = \rho \cap \mathscr{L} = \mathscr{R} \cap \mathscr{L}$$

for elements of D. Schützenberger then showed that, if S is any semigroup with an identity containing a $\mathscr{D}$-class D, say, of finite type, then S admits representations as a semigroup of matrices with entries from a group with zero determined by D. If D is both of finite type and also regular, the matrices in these representations corresponding to the elements of D are closely related to the matrices constructed by Miller and Clifford (**1**) to give a partial representation of a regular $\mathscr{D}$-class.

In certain important classes of semigroups every $\mathscr{D}$-class is of finite type. The relation λ determines a partial ordering of the $\mathscr{L}$-classes in S: if L and L' are $\mathscr{L}$-classes, $L \leqslant L'$ if $l\,\lambda\, l'$ for some $l \in L$, $l' \in L'$. Similarly ρ determines a partial ordering of the $\mathscr{R}$-classes which I also denote by $\leqslant$. Then it can be shown that the $\mathscr{D}$-class D is of finite type if and only if each $\mathscr{L}$-class in D and also each $\mathscr{R}$-class in D is a minimal element relative to these partial orderings in the set of all $\mathscr{L}$-classes and $\mathscr{R}$-classes respectively in D. Hence, from J. A. Green's Theorem 8 (**2**), that every $\mathscr{D}$-class is of finite type in a semigroup satisfying Green's minimal conditions for the relations '$\leqslant$', we see that any set of $\mathscr{L}$-classes or $\mathscr{R}$-classes respectively of S contains a minimal element.

On the other hand a regular $\mathscr{D}$-class is not necessarily of finite type. For example let S be the set of all single-valued mappings of the set I of all integers into itself with multiplication defined thus: for a, $b \in S$,

† 'de type élémentaire'.

ab is the mapping a followed by the mapping b. Then it is easy to verify that each $\mathscr{D}$-class consists of all elements of S which determine mappings of I with image sets of a given cardinal. The semigroup S is regular, so that each $\mathscr{D}$-class is also regular. Let D be the $\mathscr{D}$-class consisting of all mappings with image sets which are not finite. Each $\mathscr{L}$-class in D consists of all mappings with image sets which are a given subset of I. Let L be the $\mathscr{L}$-class determined by the image set I; then, if L' is any other $\mathscr{L}$-class in D, $L' \neq L$ and $L' \leqslant L$, and consequently D is not of finite type.

I show in this note that, by a slight modification of Schützenberger's representation theory, the restriction that D be of finite type may be removed. Any semigroup S has representations by matrices with entries from a group with zero determined by any one of its $\mathscr{D}$-classes. In the final section we consider the direct sum of the representations determined by the $\mathscr{D}$-classes of a semigroup S and show that, if S is regular, then this direct sum gives a faithful representation of S.

1. The group determined by a $\mathscr{D}$-class

We assume throughout that S contains an identity element. This assumption in fact imposes no restriction upon the validity of the results [see (**2**)] although, without the assumption, the results will need rephrasing.

Denote by L_x the $\mathscr{L}$-class and by R_x the $\mathscr{R}$-class containing x. Then we have the following fundamental lemma due to Green [(**2**) 165]:

Lemma 1 (Green). *Let $a\mathscr{R}b$ and let $b = as$ (such an element $s \in S$ necessarily exists). Then there exists an element s' in S such that the right translations*

$$\rho_s\colon\ x \to xs \quad (x \in L_a),$$

$$\rho_{s'}\colon\ y \to ys' \quad (y \in L_b)$$

are mutually inverse, one-to-one, $\mathscr{R}$-class-preserving mappings of L_a onto L_b and of L_b onto L_a, respectively.

Using this lemma Schützenberger shows that each $\mathscr{D}$-class determines a group. Denote the equivalence $\mathscr{R} \cap \mathscr{L}$ by $\mathscr{H}$.

Lemma 2 (Schützenberger). *Let H be an $\mathscr{H}$-class of S. Let*

$$T = \{t\colon\ t \in S;\ Ht = H\},$$

$$\tau = \{(t_1, t_2)\colon\ t_i \in T;\ ht_1 = ht_2 \text{ for all } h \in H\}.$$

Then τ is a congruence over the semigroup T and T/τ is a group $\Gamma(H)$

of the same cardinal as H. If K is any $\mathscr{H}$-class in the same $\mathscr{D}$-class as H, then $\Gamma(K)$ *is isomorphic to* $\Gamma(H)$.

Let
$$T' = \{t\colon t \in S;\, tH = H\},$$
$$\tau' = \{(t_1, t_2)\colon t_i \in T';\, t_1 h = t_2 h \text{ for all } h \in H\}.$$
Then τ' *is a congruence over the semigroup* T' *and* T'/τ' *is a group* $\Gamma'(H)$ *isomorphic to* $\Gamma(H)$.

$\Gamma(H)$ can be regarded as the group of all (1, 1)-mappings of H onto H determined by right multiplications by elements of S. Schützenberger shows, as it is easy to verify, that any such mapping is determined by its effect on any element of H, so that, if we choose a fixed element h_0 in H, then the mapping $h \to ht$ can be denoted unambiguously by $\phi(h_0 t)$. It follows from Lemma 1, since any two elements of H are $\mathscr{R}$-equivalent, that for any h in H there is an element $\phi(h)$ in $\Gamma(H)$ that maps h_0 into h. The multiplication of the elements of $\Gamma(H)$ is determined by the rule $\phi(h_0 t)\phi(h_0 s) = \phi(h_0 st)$. Effectively another multiplication has been defined in H and under this multiplication H is a group. When H is itself a group under the multiplication of S, then we may choose h_0 as the identity of H, and then $\Gamma(H)$ may be identified with H.

I complete this section by giving short proofs of two well-known results on $\mathscr{H}$-classes. The first theorem is Green's Theorem 7 in (**2**).

THEOREM (GREEN). *Let H be an* $\mathscr{H}$*-class of a semigroup S such that* $h_1 h_2 \in H$ *for some* h_1, h_2 *in H. Then H is a group.*

Proof. Since h_1 is $\mathscr{R}$-equivalent to $h_1 h_2$ and both h_1 and $h_1 h_2$ belong to the same $\mathscr{L}$-class, by Lemma 1, $Hh_2 = H$. Hence for any h in H, $hh_2 \in H$, and so, by the left-right dual of Lemma 1, $hH = H$ for any h in H. Similarly $Hh = H$ for any h in H; and so H is a group.

The next theorem is Theorem 3 of Miller and Clifford (**1**).

THEOREM (MILLER and CLIFFORD). *Let a and b be elements of a semigroup S. Then* $ab \in R_a \cap L_b$ *if and only if* $R_b \cap L_a$ *contains an idempotent element; if this be the case, then*
$$aH_b = H_a b = H_a H_b = H_{ab} = R_a \cap L_b.$$

Proof. Let $ab \in R_a \cap L_b$. By Lemma 1, since $a \mathscr{R} ab$, ρ_b is an $\mathscr{R}$-class-preserving (1, 1)-mapping of L_a onto L_b with an inverse $\rho_{b'}$ say. Hence in particular $bb'b = b$, so that bb' is an idempotent and $bb' \in R_b \cap L_a$, and further $H_a b = H_{ab} = R_a \cap L_b$.

The left-right dual of Lemma 1 similarly implies that $aH_b = H_{ab}$.

It now follows by an argument similar to that used to prove the previous theorem that $H_a H_b = H_{ab}$.

Conversely, suppose that $e^2 = e \in R_b \cap L_a$. Then e is a left identity for R_b [(1) Lemma 4] and so $e\mathscr{R}eb = b$. Hence, by Lemma 1, ρ_b is an $\mathscr{R}$-class-preserving (1, 1)-mapping of L_a onto L_b, so that in particular $ab \in R_a \cap L_b$.

2. The representation theorem

Let D be any $\mathscr{D}$-class of S and let $L_\kappa (\kappa \in \mathrm{K})$ and R_i $(i \in I)$ denote the $\mathscr{L}$-classes and $\mathscr{R}$-classes, respectively, of D. We may assume that $\mathrm{K} \cap I$ contains the symbol 1. Then it follows immediately from Lemma 1 and its left-right dual [(2) Theorem 1] that there exist q_κ, q'_κ $(\kappa \in \mathrm{K})$ and r_i, r'_i $(i \in I)$ belonging to S such that the mappings $x \to xq_\kappa$ and $y \to yq'_\kappa$ are mutually inverse $\mathscr{R}$-class-preserving (1, 1)-mappings between L_1 and L_κ, and the mappings $x \to r_i x$ and $y \to r'_i y$ are mutually inverse $\mathscr{L}$-class-preserving (1, 1)-mappings between R_1 and R_i. Let $H = R_1 \cap L_1$ and as in the previous section select a fixed but arbitrary h_0 in H. Then with each $s \in S$ we associate a $\mathrm{K} \times \mathrm{K}$ matrix $M(s)$ with entries from the group with zero $G(H) = \Gamma(H) \cup \{0\}$, defined as follows: $M(s) = \{m^\mu_\kappa(s)\}$ where

$$m^\mu_\kappa(s) = \begin{cases} \phi(h_0 q_\kappa s q'_\mu) & (h_0 q_\kappa s \in R_1 \cap L_\mu), \\ 0 & (\text{otherwise}). \end{cases}$$

Similarly we define the $I \times I$ matrix $M'(s) = \{m^j_i(s)\}$, where

$$m^j_i(s) = \begin{cases} \phi'(r'_i s r_j h_0) & (s r_j h_0 \in R_i \cap L_1), \\ 0 & (\text{otherwise}), \end{cases}$$

where $\phi'(h)$ denotes the element of $\Gamma'(H)$ determined by the mapping of h_0 onto h. These definitions are only a slight modification of those of Schützenberger, but the modification enables us to drop all restrictions upon D.

THEOREM 1. *The mappings $s \to M(s)$ and $s \to M'(s)$ are homomorphic mappings of S onto semigroups of matrices, where the matrices are multiplied by usual matrix multiplication and* 0 *is regarded as an additive zero in the computation of matrix products.*

Proof. We prove that $s \to M(s)$ is a homomorphic mapping. The representation $s \to M'(s)$ is the left-right dual of the representation $s \to M(s)$.

The element in the (κ, μ)-th place of the product of the matrices $M(s)M(t)$ is the formal sum

$$\sum_\nu m^\nu_\kappa(s) m^\mu_\nu(t).$$

This formal sum will make sense provided that at most one summand is non-zero, and this is true because $m^{\nu}_{\kappa}(s) \neq 0$ only if $h_0 q_\kappa s \in R_1 \cap L_\nu$ and, for a given κ, this holds for at most one ν.

Suppose firstly that $m^{\nu}_{\kappa}(s) = 0$ for all ν, so that $h_0 q_\kappa s \notin R_1$. Then $h_0 q_\kappa st \notin R_1$; for h_0, $h_0 q_\kappa st \in R_1$ implies that there exists u in S such that $h_0 q_\kappa s(tu) = h_0$. Since also $h_0 q_\kappa s = h_0(q_\kappa s)$, we have $h_0 \mathscr{R} h_0 q_\kappa s$, and $h_0 q_\kappa s \in R_1$, which is a contradiction. Hence $m^{\nu}_{\kappa}(s) = 0$ for all ν implies that $m^{\nu}_{\kappa}(st) = 0$ for all ν, and so in this case

$$\sum_{\nu} m^{\nu}_{\kappa}(s) m^{\mu}_{\nu}(t) = m^{\mu}_{\kappa}(st).$$

Secondly, suppose that $m^{\sigma}_{\kappa}(s) \neq 0$, so that $h_0 q_\kappa s \in R_1 \cap L_\sigma$. If $m^{\mu}_{\sigma}(t) \neq 0$, so that $h_0 q_\sigma t \in R_1 \cap L_\mu$, then, by Lemma 1, the mapping ρ_t is an $\mathscr{R}$-class-preserving (1, 1)-mapping of L_σ upon L_μ, and so, since $h_0 q_\kappa s \in R_1 \cap L_\sigma$, therefore $h_0 q_\kappa st \in R_1 \cap L_\mu$. Hence

$$\begin{aligned} m^{\sigma}_{\kappa}(s) m^{\mu}_{\sigma}(t) &= \phi(h_0 q_\kappa s q'_\sigma)\phi(h_0 q_\sigma t q'_\mu) = \phi(h_0 q_\kappa s q'_\sigma q_\sigma t q'_\mu) \\ &= \phi(h_0 q_\kappa st q'_\mu) = m^{\mu}_{\kappa}(st) \end{aligned}$$

since right multiplication by $q'_\sigma q_\sigma$ is the identity-mapping on L_σ to which $h_0 q_\kappa s$ belongs. A similar argument now shows that, if $m^{\mu}_{\sigma}(t) = 0$, then $m^{\mu}_{\kappa}(st) = 0$, for $m^{\mu}_{\kappa}(st) \neq 0$ implies that the mapping ρ_t is an $\mathscr{R}$-class-preserving (1, 1)-mapping of L_σ upon L_μ. Hence in all cases when $m^{\sigma}_{\kappa}(s) \neq 0$, for some σ, then

$$\sum_{\nu} m^{\nu}_{\kappa}(s) m^{\mu}_{\nu}(t) = m^{\mu}_{\kappa}(st).$$

This completes the proof of the theorem.

It is easily seen, as for the representation defined by Schützenberger, that a replacement of h_0 by some other element of H leaves the above representation unchanged. A replacement of the q_κ, q'_κ by t_κ, t'_κ, say, transforms the matrix $M(s)$ to the matrix $AM(s)A^{-1}$, where

$$A = \operatorname{diag}(a_\kappa), \qquad A^{-1} = \operatorname{diag}(a_\kappa^{-1})$$

and where $a_\kappa = \phi(h_0 t_\kappa q'_\kappa)$. A similar transformation of the $M'(s)$ results on a new choice of r_i, r'_i.

Further suppose that we replace H by $H_{i\kappa} = R_i \cap L_\kappa$. Choose as a fixed element in $H_{i\kappa}$ the element $(h_{i\kappa})_0 = r_i h_0 q_\kappa$. Then, if the elements of $\Gamma(H_{i\kappa})$ are denoted by $\phi_{i\kappa}(h_{i\kappa})$, the mapping

$$\theta\colon\ \phi(h) \to \phi_{i\kappa}(r_i h q_\kappa)$$

is an isomorphism between $\Gamma(H)$ and $\Gamma(H_{i\kappa})$.

Choose as right multipliers which effect one-to-one $\mathscr{R}$-class-preserving mappings between L_κ and L_μ the elements $p_\mu = q'_\kappa q_\mu$ and $p'_\mu = q'_\mu q_\kappa$.

With this choice let $M_{\iota\kappa}(s)$ be the matrix representation of S determined by $H_{\iota\kappa}$, the p_μ, p'_μ and the group $\Gamma(H_{\iota\kappa})$. Then it can be verified that $M_{\iota\kappa}(s) = \{M(s)\}\theta$, where by $\{M(s)\}\theta$ we mean the matrix $\{m^\nu_\mu(s)\theta\}$.

These comments, and the analogous comments about $M'(s)$, show that the representations $M(s)$ and $M'(s)$ of S are determined to within isomorphism by the $\mathscr{D}$-class D to which H belongs.

3. A faithful representation of a regular semigroup

The direct sum of a set of matrix representations of S is again a representation of S, and the question arises: *when is the direct sum of all the $\mathscr{D}$-class matrix representations of a semigroup S a faithful representation of S?* Corresponding to each $\mathscr{D}$-class in S choose representations $M(s)$ and $M'(s)$ of S as in the previous section. Denote by $\mathscr{D}(s)$ the direct sum of all the representations $M(s)$, one corresponding to each $\mathscr{D}$-class in S; similarly denote by $\mathscr{D}'(s)$ the direct sum of all the $M'(s)$; and denote by $[\mathscr{D}+\mathscr{D}'](s)$ the direct sum of $\mathscr{D}(s)$ and $\mathscr{D}'(s)$. I give in the following lemma necessary and sufficient conditions for each of the representations $\mathscr{D}(s)$, $\mathscr{D}'(s)$, and $[\mathscr{D}+\mathscr{D}'](s)$ to be faithful.

LEMMA 3. *Let*

$$Q = \{(s,t)\colon\ s, t \in S;\ xs \text{ or } xt \in R_x \text{ implies } xs = xt\}.$$

Let $$Q' = \{(s,t)\colon\ s, t \in S;\ sx \text{ or } tx \in L_x \text{ implies } sx = tx\}.$$

Then Q and Q' are congruences on S and $\mathscr{D}(s)$, $\mathscr{D}'(s)$, $[\mathscr{D}+\mathscr{D}'](s)$ are faithful representations of S if and only if, respectively, Q, Q', $Q \cap Q'$ are equal to Δ_S the identical congruence on S.

Proof. Suppose that, for some s, t in S, $\mathscr{D}(s) = \mathscr{D}(t)$, so that $M(s) = M(t)$ for each $\mathscr{D}$-class in S. Let $x \in S$ and suppose that x belongs to the $\mathscr{D}$-class D. Let D be decomposed into its $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ classes as in § 2. Suppose that $x \in H_{\iota\kappa} = R_\iota \cap L_\kappa$. Since $M(s) = M(t)$, therefore, in the notation of the previous section, $M(s)\theta = M(t)\theta$. Hence, if $xs \in R_x$, so that $xs \in R_\iota \cap L_\mu$ for some μ, then

$$(h_{\iota\kappa})_0 q'_\kappa q_\kappa s = (h_{\iota\kappa})_0 p_\kappa s \in R_\iota \cap L_\mu,$$

and $M(s)\theta = M(t)\theta$ implies that

$$(h_{\iota\kappa})_0 p_\kappa t \in R_\iota \cap L_\mu$$

and that $$\phi_{\iota\kappa}\{(h_{\iota\kappa})_0 p_\kappa s p'_\mu\} = \phi_{\iota\kappa}\{(h_{\iota\kappa})_0 p_\kappa t p'_\mu\}.$$

Hence, if $xs \in R_x$, then $p_\kappa s p'_\mu$ and $p_\kappa t p'_\mu$ determine the same right multiplication of $H_{\iota\kappa}$, and so $xp_\kappa sp'_\mu = xp_\kappa tp'_\mu$. This implies that $xp_\kappa sp'_\mu p_\mu = xp_\kappa tp'_\mu p_\mu$: that is $xp_\kappa s = xp_\kappa t$. But, by the choice of p_κ, p_κ is a right identity for $H_{\iota\kappa}$. Hence, finally, we have that $xs = xt$.

Similarly, if $xt \in R_x$, then $xt = xs$. This argument holds for any x in S; and so $(s, t) \in Q$.

Conversely, an examination of the above argument shows that it can be reversed and that $(s, t) \in Q$ implies that $\mathscr{D}(s) = \mathscr{D}(t)$.

Hence, since $\{\mathscr{D}(s)\}$ is a homomorphic image of S, Q is a congruence on S, and S/Q is isomorphic to $\{\mathscr{D}(s)\}$.

Similarly we prove that S/Q' is isomorphic to $\{\mathscr{D}'(s)\}$ and that $S/(Q \cap Q')$ is isomorphic to $\{[\mathscr{D}+\mathscr{D}'](s)\}$. The remaining assertions in the lemma then follow immediately.

As a corollary we have the following theorem:

THEOREM 2. *Let S be a regular semigroup. Then the representation $[\mathscr{D}+\mathscr{D}'](s)$ is a faithful representation of S.*

Proof. Let $(s, t) \in Q \cap Q'$. We have to show that $s = t$.

Since S is regular, s and t have inverses [see (**1**) or (**2**) for the definition of a regular semigroup]. Thus there exist elements x and y belonging to S such that

$$sxs = s, \quad xsx = x, \quad tyt = t, \quad yty = y.$$

Then, since $xsx = x$, we have that $xs \in R_x$, and so, since $(s, t) \in Q$, $xs = xt$. Again $yty = y$ implies $yt = ys$. Similarly, since $(s, t) \in Q'$, therefore $tx = sx$ and $ty = sy$. Hence

$$s = s(xs) = s(xt) = sx(tyt) = (sxt)yt = syt = (sy)t = tyt = t.$$

Thus $Q \cap Q' = \Delta_S$ and the theorem follows from Lemma 3.

Neither the representation $\mathscr{D}(s)$ nor the representation $\mathscr{D}'(s)$ is in general a faithful representation of a regular semigroup, as is shown by considering (**4**) the rectangular band $B = \{e, f, g, h\}$ of four idempotents. B is regular but neither $\mathscr{D}(s)$ nor $\mathscr{D}'(s)$ is a faithful representation of B.

Professor A. H. Clifford pointed out to me that, when S is an inverse semigroup, then the representations $\mathscr{D}(s)$ and $\mathscr{D}'(s)$ are each faithful representations of S. To see this consider again a regular semigroup S, let $(s, t) \in Q$, and let x and y be any inverses of s and t respectively. Then, as in the proof of Theorem 2, $xs = xt$ and $yt = ys$. Hence

$$s = s(xs) = s(xt) = sx(tyt) = s(xt)(yt) = s(xs)(ys) = sys,$$
$$ysy = y(ty) = y.$$

Thus y is an inverse of s. Similarly x is an inverse of t. Thus s and t are elements of S with the same set of elements of S as inverses. It follows therefore that, if S is an inverse semigroup, i.e. a regular semi-

group in which each element has a unique inverse (**5**), then $(s,t) \in Q$ implies that $s = t$. Similarly it follows that $Q' = \Delta_S$ when S is an inverse semigroup.

Finally, I give an example of a semigroup which is not regular but which is faithfully represented by the representation $\mathscr{D}(s)$. Let T be the right simple semigroup (that is, a semigroup with no proper right ideals) consisting of all $(1, 1)$-mappings α of the set I of all integers into I such that $I-\alpha(I)$ is an infinite set. Here $\alpha(I)$ denotes the image of I under α. This is the semigroup introduced by R. Baer and F. Levi (**6**). The product $\alpha\beta$ of two elements α, β in T is taken to be the mapping α followed by the mapping β. Adjoin to T the identity element 1 to form the semigroup S. Then I shall show that over the semigroup S the congruence Q coincides with Δ_S.

For let $(s,t) \in Q$, so that for any $x \in S$ xs or $xt \in R_x$ implies $xs = xt$. If $s = 1$, then in particular $xs \in R_x$ if $x = 1$, and in this case

$$t = xt = xs = 1,$$

so that $s = t$. Similarly, if $t = 1$, then $s = 1$. Suppose now that s and t both belong to T. Since T is right simple, T is an $\mathscr{R}$-class of S, and so $xs \in R_x$ for all x in T. Hence $xs = xt$ for all x in T. Suppose that $s \neq t$. Then there is an element n of I such that $s(n) \neq t(n)$. Let x be any element of T for which $n = x(n)$; since T was chosen as the set of all $(1, 1)$-mappings α of I into I such that $I-\alpha(I)$ is infinite, there certainly exist mappings x in T with this property. Then, for such an x, $xs \neq xt$ since $xs(n) \neq xt(n)$. This is a contradiction, and so $s = t$.

This completes the proof that the representation $\mathscr{D}(s)$ is a faithful representation of S. It is clear that S is not regular, for the $\mathscr{D}$-class T contains no idempotents.

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A GENERATION OF THE SYMPLECTIC GROUP

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In this paper the symplectic group $L_{2m}(G)$ is defined as the group of matrices A of $2m$ rows and columns for which

$$A^T G A = G,$$

where

$$G = \begin{bmatrix} 0 & 1 & 1 & 1 & . & . & . \\ -1 & 0 & 1 & 1 & . & . & . \\ -1 & -1 & 0 & 1 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

The elements of the matrices may belong to any field of characteristic not equal to 2. If the elements belong to $GF(p)$, then the group is isomorphic with Dickson's $SA(2m, p)$ [Dickson(**1**) 91, Frucht (**2**)].

We were led to the investigation of the generators of this group in the course of work on the geometrical loci invariant under certain groups of collineations associated with the generalized Clifford units. The particular form of the matrix G arises naturally in this work, and the matrices which are introduced as generators have their origin in certain substitution operations among sets of Clifford units.

If

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus \dots \dotplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is the canonical skew matrix, then

$$S = H^T G H,$$

where

$$H = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 1 & 0 & . & . & . \\ 0 & 1 & -1 & 0 & -1 & 0 & -1 & 0 & . & . & . \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \end{bmatrix},$$

so that generators of $L_{2m}(S)$ can be derived directly from those of $L_{2m}(G)$.

We use $\qquad \epsilon_r = \{0, 0, \dots, 1, \dots, 0\} \quad (r = 1, \dots, 2m),$

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with 1 in the rth place, for the basis for vectors of $2m$ components, and the following symbols for certain vectors of k components

$$\mathbf{o}_k = \{0, 0, \ldots, 0\},$$
$$\mathbf{u}_k = \{1, 1, \ldots, 1\},$$
$$\boldsymbol{\omega}_k = \{-1, 1, -1, \ldots, (-1)^k\}.$$

The paper is devoted to the proof of the following theorem:

THEOREM I. *Every element of $L_{2m}(G)$ can be expressed as a finite product in which each term is either Q or a matrix-linear function D^x for some value of the scalar variable x, where*

$$Q = \begin{bmatrix} \mathbf{o}_{2m-1}^T & -1 \\ I_{2m-1} & -\boldsymbol{\omega}_{2m-1} \end{bmatrix}, \qquad D^x = \begin{bmatrix} 1 & x\mathbf{u}_{2m-1}^T \\ \mathbf{o}_{2m-1} & I_{2m-1} \end{bmatrix}.$$

It is easily verified that Q and D^x belong to $L_{2m}(G)$. We have

$$Q^{2m+1} = -I,$$

and, if D is written for D^1, then, for every integer x, D^x is the xth power of D.

When the elements of the matrices belong to $GF(p)$, so that $L_{2m}(G)$ is isomorphic with $SA(2m, p)$, then the matrix-linear function D^x is always a power of D, and

$$D^p = I.$$

Theorem I then takes the form:

THEOREM II. *$SA(2m, p)$ is isomorphic with the group generated by the two matrices Q and D, where*

$$Q = \begin{bmatrix} \mathbf{o}_{2m-1}^T & -1 \\ I_{2m-1} & -\boldsymbol{\omega}_{2m-1} \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & \mathbf{u}_{2m-1}^T \\ \mathbf{o}_{2m-1} & I_{2m-1} \end{bmatrix}.$$

From Q and D^x we derive the matrices

$$D_r^x = Q^{r-1} D^x Q^{-r+1} = \begin{bmatrix} I_{r-1} & \mathbf{o}_{r-1} & O \\ -x\mathbf{u}_{r-1}^T & 1 & x\mathbf{u}_{2m-r}^T \\ O & \mathbf{o}_{2m-r} & I_{2m-r} \end{bmatrix},$$

$$P_r^x = Q^{r-1}(D^{-1} Q D^x Q^{-1} D) Q^{-r+1}$$
$$= I_{r-1} \dotplus \begin{bmatrix} 1+x & x \\ -x & 1-x \end{bmatrix} \dotplus I_{2m-r-1} \quad (r = 1, \ldots, 2m-1),$$

$$P_{2m}^x = \begin{bmatrix} \dot{I}_{2m-1} & -x\boldsymbol{\omega}_{2m-1} \\ \mathbf{o}_{2m-1}^T & 1 \end{bmatrix}.$$

Again, writing P_r for P_r^1, we have, for any integer x, P_r^x is the xth power of P_r.

For any vector $\boldsymbol{\alpha}$,

$$P_r^x(\boldsymbol{\alpha}) = (\alpha_1,\ldots,\alpha_{r-1},\bar{\alpha}_r,\bar{\alpha}_{r+1},\alpha_{r+2},\ldots,\alpha_{2m}),$$

where

$$\bar{\alpha}_r = \alpha_r + x(\alpha_r+\alpha_{r+1}),$$

$$\bar{\alpha}_{r+1} = \alpha_{r+1} - x(\alpha_r+\alpha_{r+1}) \quad (r = 1,\ldots,2m-1).$$

In relation to any given vector $\boldsymbol{\alpha}$ for which

$$\alpha_r+\alpha_{r+1} \neq 0$$

we define P_r^* as

$$P_r^* = P_r^x \quad \text{for the value } x = \alpha_{r+1}/(\alpha_r+\alpha_{r+1}),$$

so that

$$P_r^*(\boldsymbol{\alpha}) = (\alpha_1,\ldots,\alpha_{r-1},\alpha_r+\alpha_{r+1},0,\alpha_{r+2},\ldots,\alpha_{2m}). \qquad (P_r^*)$$

We have also, when $\alpha_r = 0$,

$$P_r(\alpha_1,\ldots,\alpha_{r-1},0,\alpha_{r+1},\ldots,\alpha_{2m}) = (\alpha_1,\ldots,\alpha_{r-1},\alpha_{r+1},0,\alpha_{r+2},\ldots,\alpha_{2m}). \qquad (P_r)$$

Almost the whole of the reduction of a matrix A to a product of terms Q and D^x is performed with the use of the matrices P_r^* and P_r. We proceed first to find a matrix R_1 such that

$$R_1\boldsymbol{\alpha} = \boldsymbol{\epsilon}_1,$$

$\boldsymbol{\alpha}$ being supposed to be the first column of the matrix A.

(ia) Suppose that

$$\sum_{t=0}^{s}\alpha_{2m-t} \neq 0, \quad \text{for all } s = 1,\ldots,r-1,$$

while

$$\sum_{t=0}^{r}\alpha_{2m-t} = 0.$$

Then

$$P_{2m-1}^*(\boldsymbol{\alpha}) = (\alpha_1,\ldots,\alpha_{2m-2},\alpha_{2m-1}+\alpha_{2m},0),$$

$$P_{2m-2}^*P_{2m-1}^*(\boldsymbol{\alpha}) = (\alpha_1,\ldots,\alpha_{2m-3},\alpha_{2m-2}+\alpha_{2m-1}+\alpha_{2m},0,0),$$

$$P_{2m-r+1}^*P_{2m-r+2}^*\ldots P_{2m-1}^*(\boldsymbol{\alpha}) = (\alpha_1,\ldots,\alpha_{2m-r},-\alpha_{2m-r},\mathbf{o}_{r-1}).$$

Because the sum of the last two non-zero components is zero, we cannot proceed further with this type of reduction.

(ib) Suppose that

$$\alpha_{2m-s}+\alpha_{2m-s+1} = 0 \quad \text{for all } s = 1,\ldots,r-1,$$

while

$$\alpha_{2m-r}+\alpha_{2m-r+1} \neq 0,$$

so that

$$\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_{2m-r},-\alpha_{2m-r+1}\boldsymbol{\omega}_r).$$

Then

$$P_{2m-r}^*(\boldsymbol{\alpha}) = (\alpha_1,\ldots,\alpha_{2m-r-1},\alpha_{2m-r}+\alpha_{2m-r+1},0,\alpha_{2m-r+1}\boldsymbol{\omega}_{r-1}),$$

and we can proceed with the reduction of

$$(\alpha_1,\ldots,\alpha_{2m-r-1},\alpha_{2m-r}+\alpha_{2m-r+1})$$

using only matrices from the set $P_1^*, \ldots, P_{2m-r-1}^*$, none of which affects the remaining r components of the vector.

(ii) The result of a sequence of operations of the forms (ia) and (ib) is to reduce $\boldsymbol{\alpha}$ to the form

$$\boldsymbol{\beta} = (\beta_1 \boldsymbol{\omega}_{s_1}, \mathbf{o}_{t_1}, \beta_2 \boldsymbol{\omega}_{s_2}, \mathbf{o}_{t_2}, \ldots, \beta_i \boldsymbol{\omega}_{s_i}, \mathbf{o}_{t_i}).$$

Use operations P_r to move all zeros to the right. Then further applications of the operations P_r^* in relation to pairs of successive terms whose sum is not zero, and of operations P_r to move zeros to the right, will result in the reduction of $\boldsymbol{\alpha}$ to

$$\boldsymbol{\gamma} = (\gamma \boldsymbol{\omega}_n, \mathbf{o}_{2m-n}),$$

where $\gamma \neq 0$, since A is not singular.

The reduction from this form presents some unexpected complications. In view of its application at a later stage we effect first the reduction in the case in which $\gamma = -1$ and n is odd.

(iii) Let
$$\boldsymbol{\eta} = (-\boldsymbol{\omega}_{2\nu+1}, \mathbf{o}_{2m-2\nu-1}).$$
Then
$$P_{2\nu+1}(\boldsymbol{\eta}) = (-\boldsymbol{\omega}_{2\nu}, 2, -1, \mathbf{o}_{2m-2\nu-2}),$$
$$P_1^{\frac{1}{2}} P_2^{2} P_3^{\frac{1}{2}} P_4^{2} \ldots P_{2\nu}^{2} P_{2\nu+1}(\boldsymbol{\eta}) = (2, \mathbf{o}_{2\nu}, -1, \mathbf{o}_{2m-2\nu-2}).$$
We can now move all zeros to the right by operations P_r and so obtain
$$\boldsymbol{\eta}' = (2, -1, \mathbf{o}_{2m-2}).$$
Finally
$$P_1^{-1}(\boldsymbol{\eta}') = \boldsymbol{\epsilon}_1.$$
It is to be noted that this operation cannot be carried out if the field is of characteristic 2, and for that reason the theorem cannot be proved (at least in this form) for such fields.

(iv) Return now to the general case
$$\boldsymbol{\gamma} = (\gamma \boldsymbol{\omega}_n, \mathbf{o}_{2m-n})$$
in which γ is unrestricted, and suppose first that n is even. Then
$$D_{2\rho}^{-1}(\gamma \boldsymbol{\omega}_{2\rho}, \mathbf{o}_{2m-2\rho}) = (\gamma \boldsymbol{\omega}_{2\rho-1}, \mathbf{o}_{2m-2\rho+1}).$$
If $\gamma = -1$, this vector can be reduced as in (iii).

(v) Suppose then that $\boldsymbol{\alpha}$ has been reduced to
$$\boldsymbol{\gamma} = (\gamma \boldsymbol{\omega}_{2\rho-1}, \mathbf{o}_{2m-2\rho+1}),$$
where
$$\gamma + 1 \neq 0.$$
Then
$$\boldsymbol{\gamma}' = P_{2\rho-1}^{1/\gamma}(\boldsymbol{\gamma}) = (\gamma \boldsymbol{\omega}_{2\rho-2}, -\gamma-1, 1, \mathbf{o}_{2m-2\rho}),$$
$$\boldsymbol{\gamma}'' = P_1^* P_2^* \ldots P_{2\rho-2}^*(\boldsymbol{\gamma}') = (-\gamma-1, \mathbf{o}_{2\rho-2}, 1, \mathbf{o}_{2m-2\rho}).$$
Finally
$$D_1^{\gamma+1}(\boldsymbol{\gamma}'') = (\mathbf{o}_{2\rho-1}, 1, \mathbf{o}_{2m-2\rho}),$$
and this can be reduced by operations P_r to $\boldsymbol{\epsilon}_1$.

Thus, whatever the first column of A may be, we can find a matrix R_1, a product of matrices P_r^x and D_r^x and therefore of matrices Q and D^x, such that $A_1 = R_1 A$ is a matrix with first column ϵ_1.

(vi) Suppose that matrices $R_1, R_2,..., R_{k-1}$ $(2 \leqslant k \leqslant 2m-1)$ have been found such that

$$A_{k-1} = R_{k-1} R_{k-2} ... R_1 A = (\epsilon_1, \epsilon_2,..., \epsilon_{k-1}, \varkappa, \lambda_1,..., \lambda_{2m-k}).$$

We wish to find a matrix R_k, a product of matrices Q and D^x, such that the first k columns of $R_k A_{k-1}$ are $\epsilon_1,..., \epsilon_{k-1}, \epsilon_k$. We find in fact R_k as a product of terms $P_k^x, ..., P_{2m}^x$ (for various values of x); each of these clearly leaves $\epsilon_1,..., \epsilon_{k-1}$ unchanged.

Since
$$A_{k-1}^T G A_{k-1} = G,$$
we must have
$$\epsilon_1^T G\varkappa = \epsilon_2^T G\varkappa = ... = \epsilon_{k-1}^T G\varkappa = 1,$$
and therefore

$$\begin{aligned}\kappa_2+\kappa_3+...+\kappa_{k-1} &= -\kappa_1+\kappa_3+\kappa_4+...+\kappa_{k-1}\\ &= -\kappa_1-\kappa_2+\kappa_4+...+\kappa_{k-1} = ... = -\kappa_1-\kappa_2-...-\kappa_{k-2}\\ &= 1-(\kappa_k+\kappa_{k+1}+...+\kappa_{2m}),\end{aligned}$$

i.e.
$$\kappa_1+\kappa_2 = \kappa_2+\kappa_3 = ... = \kappa_{k-2}+\kappa_{k-1} = 0,$$
say
$$\kappa_i = (-1)^i \kappa \quad (i = 1,..., k-1).$$
Thus
$$\varkappa = (\kappa\omega_{k-1}, \kappa_k, ..., \kappa_{2m}).$$

The first step in the reduction is to obtain zeros for the first $k-1$ terms: we have

$$\varkappa' = P_{2m}^{\kappa/\kappa_{2m}}(\varkappa) = (o_{k-1}, \kappa_k+(-1)^k\kappa, \kappa_{k+1}+(-1)^{k+1}\kappa, ..., \kappa_{2m}).$$

We may now reduce $\varkappa'$ by steps corresponding to (i) and (ii) to the form
$$\varkappa'' = (o_{k-1}, \kappa''\omega_\rho, o_{2m-k-\rho+1}).$$

These steps involve the use only of operations from the set $P_k^x, P_{k+1}^x, ..., P_{2m-1}^x$ for various values of x.

Applying the condition that $\varkappa''$ is the kth column of a matrix belonging to $L_{2m}(G)$ whose first $k-1$ columns are $\epsilon_1,..., \epsilon_{k-1}$, we find that

$$\rho \text{ is odd}, \qquad \kappa'' = -1,$$

and we may therefore use operations of type (iii), again involving only $P_k^x,..., P_{2m}$, to reduce $\varkappa''$ to ϵ_k.

That is, we have found the required matrix R_k.

(vii) Since we have already obtained a matrix R_1 such that the first column of $R_1 A$ is ϵ_1, we can obtain successively matrices $R_2,..., R_{2m-1}$

which reduce the next $2m-2$ columns to the required forms, i.e. matrices such that

$$A_{2m-1} = R_{2m-1} R_{2m-2} \dots R_1 A = (\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \dots, \boldsymbol{\epsilon}_{2m-1}, \boldsymbol{\mu}).$$

Since
$$A_{2m-1}^T G A_{2m-1} = G,$$
we have
$$\mu_i = (-1)^i \mu \quad (i = 1, \dots, 2m-1)$$
and
$$\mu_{2m} = 1,$$
so that
$$\boldsymbol{\mu} = (\mu \boldsymbol{\omega}_{2m-1}, 1).$$
Finally
$$P_{2m}^{\mu}(\boldsymbol{\mu}) = \boldsymbol{\epsilon}_{2m},$$
and therefore
$$P_{2m}^{\mu} A_{2m-1} = I,$$
i.e.
$$A = (P_{2m}^{\mu} R_{2m-1} R_{2m-2} \dots R_1)^{-1},$$
where each term is the product of a finite number of terms P_r^x and D_r^x, i.e. of powers of Q and values of the linear matrix function D^x.

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ON THE INTEGRAL EQUATION FOR THE FINITE DAM

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1. Introduction

IN considering a model for the dam of finite capacity k, Moran (**2**) makes the following assumptions: (i) the inputs X_t $(t = 0, 1, 2,...)$ which flow into the dam in the yearly intervals $(t, t+1)$ are independently and identically distributed; (ii) if Z_t $(\leqslant k)$ is the storage at time t before the input X_t flows into the dam, then, for $Z_t+X_t > k$, an amount Z_t+X_t-k will overflow, but, for $Z_t+X_t \leqslant k$, there will be no overflow; the dam will then contain a quantity k or Z_t+X_t, whichever is the less; (iii) at time $t+1$, the amount of water released is m if $Z_t+X_t > m$ or Z_t+X_t if $Z_t+X_t \leqslant m$, where $m < k$. It is then clear $\{Z_t+X_t\}$ and $\{Z_t\}$ are both Markov chains, and, for a given probability distribution of the input X_t, their stationary distributions may be studied. In the case where the probability distribution of X_t is of the continuous type with the cumulative distribution function (c.d.f.) $G(x)$, so that

$$\Pr\{x < X_t \leqslant x+dx\} = dG(x) \quad (0 < x < \infty), \tag{1}$$

the c.d.f. $H(y)$ of the stationary distribution of the dam content Z_t+X satisfies the integral equation

$$H(y) = \begin{cases} -\int\limits_m^{m+y} H(t)\,dG(m+y-t) & (y < k-m), \\ G(y-k+m) - \int\limits_m^{k} H(t)\,dG(m+y-t) & (y \geqslant k-m) \end{cases} \tag{2}$$

together with $H(\infty) = 1$ [cf. Moran (**2**), where the equation is written in terms of the frequency function of the stationary distribution].

It might be useful to solve the integral equation (2) for the important class of input distributions

$$dG(x) = \frac{\mu^p}{(p-1)!} e^{-\mu x} x^{p-1}\,dx \quad (0 < x < \infty), \tag{3}$$

where $\mu > 0$ and $p = 1, 2,...$. Moran (**3**) obtained an exact solution (although by a different technique) for the case of the negative exponential input [corresponding to $p = 1$ in (3)], and also in (**4**) for the general gamma-type input when $k \to \infty$ and the release is continuous. Gani and Prabhu (**1**) studied the case where $H(y)$ is of an approximate

gamma-type, and found that the associated $G(x)$ is also of approximate gamma-type, for $k \to \infty$ but m discrete. In this paper I obtain an exact solution for the integral equation (2) when the input distribution is of the general gamma-type (3).

2. Solution of the integral equation in the case of a gamma-type input

Let us consider first the case of a negative exponential input,

$$dG(x) = \mu e^{-\mu x}\,dx \quad (0 < x < \infty;\ \mu > 0). \tag{4}$$

Moran's solution (3) for the frequency function $h(y)$ of the stationary distribution of the dam content is given by the equation

$$h(k-y) = c\sum_{q=1}^{N+1}\frac{\mu^q(-1)^{q-1}}{(q-1)!}[(y-\langle q-1\rangle m)^{q-1}\exp\{\mu(y-\langle q-1\rangle m)\}- \\ -(y-qm)^{q-1}\exp\{\mu(y-qm)\}],$$

where $\exp x = e^x$ if $x \geqslant 0$ or 0 if $x < 0$, $k = (N+1)m+U$, $0 < U < m$, and c is the normalizing constant. This can be written as

$$h(k-y) = \begin{cases} c\mu e^{\mu y} & (-\infty < y \leqslant m), \\ ce^{\mu y}\left\{\mu\sum\limits_{q=0}^{n}\dfrac{(-\lambda)^q}{q!}(y-qm)^q+\sum\limits_{q=1}^{n}\dfrac{(-\lambda)^q}{(q-1)!}(y-qm)^{q-1}\right\} & \\ \qquad (nm < y \leqslant (n+1)m;\ n = 1,2,\ldots,N+1), \end{cases}$$

where $\lambda = \mu e^{-\mu m}$. From this we obtain

$$1-H(k-y) = \int\limits_{-\infty}^{y} h(k-t)\,dt = \begin{cases} ce^{\mu y} & (-\infty < y \leqslant m), \\ ce^{\mu y}\sum\limits_{q=0}^{n}\dfrac{(-\lambda)^q}{q!}(y-qm)^q & \\ \qquad (nm < y \leqslant (n+1)m). \end{cases} \tag{5}$$

This suggests the substitution

$$\Phi(y) = e^{-\mu y}\{1-H(k-y)\} \quad (-\infty < y \leqslant k) \tag{6}$$

in the integral equation (2) when the input is of the general gamma-type (3); the equation then reduces to

$$\Phi(y) = e^{-\mu k}\sum_{r=0}^{p-1}\frac{\mu^r}{r!}(k-y)^r+ \\ +\begin{cases} \mu^p e^{-\mu m}\displaystyle\int\limits_{0}^{k-m}\Phi(t)\frac{(t-y+m)^{p-1}}{(p-1)!}\,dt & (-\infty < y \leqslant m), \\ \mu^p e^{-\mu m}\displaystyle\int\limits_{y-m}^{k-m}\Phi(t)\frac{(t-y+m)^{p-1}}{(p-1)!}\,dt & (m < y \leqslant k), \end{cases} \tag{7}$$

which is a mixture of both Fredholm and Volterra types of integral equation, with the kernel $(t-y+m)^{p-1}/(p-1)!$, but owing to the presence of m in the lower limit of the integral on the right-hand side (for $m < y \leqslant k$) the known methods for solving such equations are not directly applicable. However, we note that the kernel is resolvable: in fact, we have

$$\frac{(t-y+m)^{p-1}}{(p-1)!} = \sum_{r=0}^{p-1} \frac{(-1)^r}{r!\,(p-r-1)!} y^r (t+m)^{p-r-1}.$$

Let us put

$$e^{-\mu k} \sum_{r=0}^{p-1} \frac{\mu^r}{r!}(k-y)^r + \mu^p e^{-\mu m} \int_0^{k-m} \Phi(t) \frac{(t-y+m)^{p-1}}{(p-1)!}\, dt = \sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r,$$

where

$$\alpha_r = e^{-\mu k}(-\mu)^r \sum_{s=0}^{p-r-1} \frac{(\mu k)^s}{s!} + \mu^p e^{-\mu m}(-1)^r \int_0^{k-m} \Phi(t) \frac{(t+m)^{p-r-1}}{(p-r-1)!}\, dt$$

$$(r = 0, 1, \ldots, p-1). \quad (8)$$

Then the equation (7) can be written as

$$\Phi(y) = \begin{cases} \displaystyle\sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r \quad (-\infty < y \leqslant m), & (9) \\ \displaystyle\sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r - \lambda \int_0^{y-m} \Phi(t) \frac{(y-m-t)^{p-1}}{(p-1)!}\, dt \quad (m < y \leqslant k), & (10) \end{cases}$$

where $\lambda = (-1)^{p-1}\mu^p e^{-\mu m}$. It is seen that the integral on the right-hand side of (10) involves $\Phi(t)$ in the range $(0, y-m)$; this enables us to solve for $\Phi(y)$ successively for the ranges $(m, 2m)$, $(2m, 3m)$,... in terms of the unknown constants $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$. For instance, let $m < y \leqslant 2m$; then

$$\Phi(y) = \sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r - \lambda \int_0^{y-m} \sum_{r=0}^{p-1} \frac{\alpha_r t^r (y-m-t)^{p-1}}{r!\,(p+1)!}\, dt$$

$$= \sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r - \lambda \sum_{r=0}^{p-1} \frac{\alpha_r (y-m)^{p+r}}{(p+r)!}.$$

This suggests the general expression

$$\Phi(y) = \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{n} (-\lambda)^q \frac{(y-qm)^{qp+r}}{(qp+r)!}$$

$$(nm < y \leqslant (n+1)m;\ n = 1, 2, \ldots, N+1). \quad (11)$$

To prove that this, in fact, is the solution to (9) we use the method of induction. Assume that $\Phi(y)$ is given by the above expression in the range $0 < y \leqslant (n+1)m$. Then for $(n+1)m < y \leqslant (n+2)m$ we have, from (9),

$$\Phi(y) = \sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r - \lambda \int_0^{y-m} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{n} (-\lambda)^q \frac{(t-qm)^{qp+r}(y-m-t)^{p-1}}{(qp+r)!\,(p-1)!}\, dt$$

$$= \sum_{r=0}^{p-1} \frac{\alpha_r}{r!} y^r + \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{n} (-\lambda)^{q+1} \frac{\{y-(q+1)m\}^{qp+p+r}}{(qp+p+r)!}$$

$$= \sum_{r=0}^{p-1} \frac{\alpha_r}{r!} \sum_{q=0}^{n+1} (-\lambda)^q \frac{(y-qm)^{qp+r}}{(qp+r)!}.$$

Hence the result follows.

It remains to evaluate the unknown constants $\alpha_0, \alpha_1,..., \alpha_{p-1}$ occurring in the expressions (9) and (10), so that $\Phi(y)$ will then be completely known for the entire range $(-\infty < y \leqslant k)$. These, however, are determined by (8). We have

$$(-1)^r \mu^p e^{-\mu m} \int_0^{k-m} \Phi(t) \frac{(t+m)^{p-r-1}}{(p-r-1)!}\, dt$$

$$= (-1)^{p+r-1} \lambda \sum_{s=0}^{p-1} \alpha_s \sum_{q=0}^{N} (-\lambda)^q \int_{qm}^{k-m} \frac{(t-qm)^{qp+s}(t+m)^{p-r-1}}{(qp+s)!\,(p-r-1)!}\, dt$$

$$= \lambda \sum_{s=0}^{p-1} d_{rs} \alpha_s,$$

where

$$d_{rs} = (-1)^{p+r-1} \sum_{q=0}^{N} (-\lambda)^q \int_{qm}^{k-m} \frac{(t-qm)^{qp+s}(t+m)^{p-r-1}}{(qp+s)!\,(p-r-1)!}\, dt$$

$$(r, s = 0, 1,..., p-1). \quad (12)$$

Then the equations (8) can be written as

$$\alpha_r - \lambda \sum_{s=0}^{p-1} d_{rs} \alpha_s = (-\mu)^r e^{-\mu k} \sum_{s=0}^{p-r-1} \frac{(\mu k)^s}{s!} \quad (r = 0, 1,..., p-1), \quad (13)$$

which are p linear equations in the p unknowns $\alpha_0, \alpha_1,..., \alpha_{p-1}$ and have a unique solution provided that the determinant $|1-\lambda D|$ does not vanish: that is, provided that λ^{-1} is not a characteristic root of the matrix

$\|d_{rs}\| = D$. Assuming this condition to be satisfied, we have the solution

$$\Phi(y) = \begin{cases} \sum_{r=0}^{p-1} \frac{\alpha^r}{r!} y^r \quad (-\infty < y \leqslant m), \\ \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{n} (-\lambda)^q \frac{(y-qm)^{qp+r}}{(qp+r)!} \\ \qquad (nm < y \leqslant (n+1)m;\, n = 1,2,\ldots, N+1). \end{cases} \tag{14}$$

3. Stationary distributions of the dam content and the dam storage

I proceed to obtain the stationary distributions of the dam content and the dam storage. From (6) we have

$$H(y) = 1 - e^{\mu(k-y)}\Phi(k-y),$$

which gives, if we take $k = (N+1)m$ for convenience,

$$H(y) = \begin{cases} 1 - e^{\mu(k-y)} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{N-s} (-\lambda)^q \frac{(k-y-qm)^{qp+r}}{(qp+r)!} \\ \qquad (sm \leqslant y < (s+1)m;\, s = 0,1,\ldots, N-1), \\ 1 - e^{\mu(k-y)} \sum_{r=0}^{p-1} \frac{\alpha_r}{r!}(k-y)^r \quad (y \geqslant Nm) \end{cases} \tag{15}$$

for the stationary distribution of the dam content $Z_t + X_t$. From (14) we note that, for large negative y, $\Phi(y)$ behaves like y^{p-1}, so that, from (6), $H(\infty) = 1$, as required. For the dam storage Z_t we have the relations

$$\Pr\{Z_{t+1} = 0\} = \Pr\{Z_t + X_t \leqslant m\},$$
$$\Pr\{Z_{t+1} = k-m\} = \Pr\{Z_t + X_t > k\},$$
$$\Pr\{0 < Z_{t+1} \leqslant z\} = \Pr\{m < Z_t + X_t \leqslant m+z\} \quad (z < k-m)$$

as a consequence of the release rule. From these relations we see that the stationary distribution of Z_t has discontinuities at $z = 0$ and $z = k-m$ given respectively by

$$\left.\begin{aligned} F(0) &= H(m) = 1 - e^{\mu(k-m)} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{N-1} (-\lambda)^q \frac{\{(N-q)m\}^{qp+r}}{(qp+r)!} \\ P_{k-m} &= 1 - H(k) = \alpha_0 \end{aligned}\right\}, \tag{16}$$

while, in the range $0 < z < k-m$, its c.d.f. is given by

$$\left.\begin{aligned} F(z) &= H(m+z) \\ &= 1 - e^{\mu(k-z-m)} \sum_{r=0}^{p-1} \alpha_r \sum_{q=0}^{N-s-1} \frac{\{(N-q)m-z\}^{qp+r}}{(qp+r)!} \\ &\qquad (sm < z \leqslant (s+1)m;\, s = 0,1,\ldots, N-1) \end{aligned}\right\}. \tag{17}$$

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COMPLEX EXTENSIONS

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Introduction

THE object of this paper is to show how real analytic structures of differential geometry can be embedded in similar complex structures in such a way that they form the real parts of them. A familiar example of this kind of technique is the use of imaginary points with complex coordinates in real projective geometry. Another example is provided by almost-complex structures where it is usual to take complex coordinates locally to clarify the integrability conditions.

The paper is in three parts. The first shows how analytic functions and regular mappings can be extended from the real to the complex field and it leads to the definition of complex extension of a manifold. It concludes with some properties and with brief proofs of two theorems: one shows the relationship between 'complex extension' and 'real submanifold' as defined by Ehresmann, and the other states that the germ of complex extension is unique. The second part is devoted to a proof of the theorem that any analytic real manifold with a countable base of open sets admits a complex extension. It is shown earlier that this is equivalent to a theorem stated without proof by Ehresmann. It was however obtained independently by the author, and a detailed proof is given here partly because I think that the theorem is important and partly because it may have been assumed that the proof is more obvious than in fact it is. The final part is concerned with the extension of manifolds carrying some additional analytic structure.

1. Definitions and basic results

1.1. *Notation.* If f is a mapping of a subset of a space onto a subset of another space, it is convenient to consider these subsets as depending on f by writing $f\colon U(f) \to V(f)$. If f, g are two such mappings and if $U(f)$, $V(g)$ are in the same space, the most useful law of composition is $(f,g) \to fg$ where fg is the onto mapping

$$fg\colon \overset{-1}{g}\{U(f) \cap V(g)\} \to f\{U(f) \cap V(g)\}.$$

The real number space $\mathbf{R}^n$ is always considered as the real part of the complex number space $\mathbf{C}^n$ in the natural way. Let the modulus $|\mathbf{z}|$ of a complex vector $\mathbf{z}$ of $\mathbf{C}^n$ be given by $|\mathbf{z}| = \sqrt{\left(\sum_{i=1}^{n} z_i \bar{z}_i\right)}$, where the z_i are the components of $\mathbf{z}$.

1.2. *Extensions of functions.* This section may be read in conjunction with chapter ii § 2 of Bochner and Martin [(**1**) 33]. A complex-valued analytic function g defined on an open set $U(g)$ of $\mathbf{C}^n$ is said to be a *complex extension* of its restriction f to $\mathbf{R}^n \cap U(g)$. For example, if f is a real-valued or complex-valued function defined by an absolutely convergent power series $F(\mathbf{x}-\mathbf{a})$ in the components of $\mathbf{x}-\mathbf{a}$ (where $\mathbf{x}$ is a variable point of $\mathbf{R}^n$ and $\mathbf{a}$ is a fixed point) and $U(f)$ is the open ball of convergence given by $|\mathbf{x}-\mathbf{a}| < k$, then $F(\mathbf{z}-\mathbf{a})$ is convergent in the open ball of $\mathbf{C}^n$ given by $|\mathbf{z}-\mathbf{a}| < k$ and defines a complex extension g. It should be noted that a complex extension is not determined uniquely by its real part f because infinitely many restrictions and analytic continuations of a given complex extension are also complex extensions.

Suppose, however, that g is a complex extension of f and that $U = U(g)$ is connected to $\mathbf{R}^n$, i.e. each connected component of U contains a point of $U(f)$, then g is the only complex extension of f defined on U: it is determined uniquely by f and U.

To prove this we note that analytic functions defined on a given connected open set are uniquely determined by their power series at a single point. Let U' be a connected component of U and let $\mathbf{x}$ be a point of $U' \cap \mathbf{R}^n$. The coefficients of the power series for f and g at $\mathbf{x}$ must be the same. Hence f determines g uniquely in U' and in every other connected component. Moreover, the processes of differentiating f and g at $\mathbf{x}$ are formally the same when expressed in terms of the power series, which shows that the derivatives of a complex extension are complex extensions of the derivatives of the real part.

The existence of complex extensions for a general analytic function f is now easily verified. Each point of $U(f)$ has an open ball $U_\mathbf{x}$ centred at $\mathbf{x}$ and contained in $U(f)$ such that the restriction $f_\mathbf{x}$ of f to $U_\mathbf{x}$ is given by an absolutely convergent power series. Hence $f_\mathbf{x}$ can be extended to give a complex function $g_\mathbf{x}$ such that $U(g_\mathbf{x})$ is the complex open ball centred at $\mathbf{x}$ meeting $\mathbf{R}^n$ in $U_\mathbf{x}$. The functions $g_\mathbf{x}$ fit together to form a function g on

$$\bigcup_{\mathbf{x} \in U(f)} U(g_\mathbf{x})$$

if, given any two points $\mathbf{x}$, $\mathbf{y}$ of $U(f)$, the restrictions of $g_\mathbf{x}$ and $g_\mathbf{y}$ to

$U(g_x) \cap U(g_y)$ are the same. This is true because $U(g_x) \cap U(g_y)$ is connected to $\mathbf{R}^n$ and the restrictions are both complex extensions of the restriction of f to $U_x \cap U_y$.

1.3. *Extension of local automorphisms.* Let Λ_n^ω be the pseudo-group of analytic, regular homeomorphisms between open sets of $\mathbf{R}^n$ and let Λ_n^c be the corresponding pseudogroup for $\mathbf{C}^n$ [Ehresmann (3) 139]. Consider the subset Λ_n^e of Λ_n^c consisting of those mappings which give members of Λ_n^ω when restricted to $\mathbf{R}^n$ and whose inverses do likewise: that is,

$$\Lambda_n^e = \{g : g \in \Lambda_n^c,\ g|_{\mathbf{R}^n} \in \Lambda_n^\omega,\ \overset{-1}{g}|_{\mathbf{R}^n} \in \Lambda_n^\omega\}.$$

A member of Λ_n^e will be called a *complex extension* of its restriction to $\mathbf{R}^n$. The conventional mapping of the empty set in $\mathbf{R}^n$ admits complex extensions of the form $g \in \Lambda_n^c$ such that $U(g) \cap \mathbf{R}^n = \emptyset$.

The essential properties of Λ_n^e are contained in the next three propositions.

Proposition 1. *The set Λ_n^e forms a pseudogroup of transformations of $\mathbf{C}^n$.*

In fact, the three conditions for Λ_n^e to be a pseudogroup are easily proved, and will only be quoted:

(*a*) The sets mapped by members of Λ_n^e are a system of open sets: they are, in fact, the open sets of $\mathbf{C}^n$ because the identity mapping of each is a member.

(*b*) If g is a (1–1) mapping of $U(g) = \bigcup_\alpha U_\alpha$, where each U_α is open in $\mathbf{C}^n$, then g belongs to Λ_n^e if and only if its restriction to each U_α belongs to Λ_n^e.

(*c*) The inverse of a member of Λ_n^e is a member as is the composition of any two.

Proposition 2. *If g is a member of Λ_n^e defined on U which is connected to $\mathbf{R}^n$, then g is uniquely determined by its restriction f to $\mathbf{R}^n$ and the set U.*

This follows from the corresponding result for functions because the n complex functions which define g must be complex extensions of those which define f.

Proposition 3. *Every member f of Λ_n^ω has a complex extension g (and therefore infinitely many).*

A member f of Λ_n^ω is an analytic mapping of an open set $U(f)$ of $\mathbf{R}^n$

into $\mathbf{R}^n$ and can be written $f = (f_1, f_2, \ldots, f_n)$, where the f_i $(i = 1, 2, \ldots, n)$ are real analytic functions on $U(f)$. Let g_i be complex extensions of the f_i. For each point $\mathbf{x}$ of $U(f)$, let $U_{\mathbf{x}}$ be some open neighbourhood of $\mathbf{x}$ in $\bigcap_{i=1}^{n} U(g_i)$ and let $g_{\mathbf{x}}$ be the mapping of $U_{\mathbf{x}}$ into $\mathbf{C}^n$ defined by restricting the functions g_i. I shall prove that, if each $U_{\mathbf{x}}$ is chosen carefully, then the mapping g formed by fitting together the mappings $g_{\mathbf{x}}$, as $\mathbf{x}$ varies over $U(f)$, is a complex extension of f. Note that the Jacobian of the g_i takes the same value as the Jacobian of the f_i at $\mathbf{x}$ and is therefore non-zero. It follows from the implicit-function theorem [(**1**) 39] that the $U_{\mathbf{x}}$ can be chosen such that each $g_{\mathbf{x}}$ is regular. Let us choose the $U_{\mathbf{x}}$ such that $g_{\mathbf{x}}$ is regular and that $g_{\mathbf{x}}(U_{\mathbf{x}})$ is an open ball centred at $g_{\mathbf{x}}(\mathbf{x})$. Then, for two points $\mathbf{x}$ and $\mathbf{y}$ of $U(f)$, we have that $g_{\mathbf{x}}(U_{\mathbf{x}}) \cap g_{\mathbf{y}}(U_{\mathbf{y}})$ is connected to $\mathbf{R}^n$. However, the restrictions of $\overset{-1}{g_{\mathbf{x}}}$ and $\overset{-1}{g_{\mathbf{y}}}$ to this open set are both complex extensions of a restriction of $\overset{-1}{f}$, and so Proposition 2 ensures that they are the same. This completes the proof because g is locally regular and one-to-one.

1.4. *Extension of manifolds.* Let M be a real analytic manifold defined by a complete atlas $\mathscr{A}$ of a Hausdorff space onto $\mathbf{R}^n$ compatible with Λ_n^ω [(**3**) 139]. A complex extension of M is a complex analytic manifold N which contains M as a subset and which has a subatlas $\mathscr{B}$ compatible and complete with respect to Λ_n^c such that $\mathscr{A}$ is the set of restrictions of members of $\mathscr{B}$ to $\mathbf{R}^n$. The embedding of M in N will then be analytic and proper.

If N, N' are complex extensions of M, M', respectively, and if f is an analytic mapping of M into M', then there exists an analytic mapping g of an open neighbourhood U of M in N into N' such that f is the restriction of g to M. To prove this, let U_α be a collection of open sets in N which cover M and on which mappings g_α are defined by extending restrictions of f. The system U_α has a locally finite refinement and therefore also a refinement such that the intersection of every pair is connected to M. The restrictions of the mappings g_α to open sets of the refinement agree in the overlaps and will fit together to form g.

In the case when f is locally-regular, g can be chosen locally-regular. In particular, if f is an isomorphism between M and M', the extension process can be applied to f and $\overset{-1}{f}$, as in Proposition 3, making g an isomorphism of U on $g(U)$.

The next theorem is an immediate consequence of this.

UNIQUENESS THEOREM.† *If N, N' are complex extensions of M, there exists a neighbourhood of M in N isomorphic with a neighbourhood of M in N'.*

The following theorem relates the complex extension of a manifold with Ehresmann's definition of real submanifold [(2) 417]:

THEOREM. *If N is a complex manifold containing M as a proper, real, closed, analytic, submanifold and if* $\dim M = \frac{1}{2}$ real $\dim N$, *then N has a unique subatlas B which makes it a complex extension of M.*

Consider the coordinate mappings f of the manifold N such that $V(f)$ does not intersect M and $U(f)$ does not intersect $\mathbf{R}^n$. These mappings form an atlas for $N-M$.

For a given point y of M, let g and h be coordinate mappings of N and M respectively giving coordinate neighbourhoods $V(g)$ and $V(h)$ of y. Since M is properly embedded, we can choose g such that $V(g) \cap M \subset V(h)$, and, since the embedding is analytic, $\overset{-1}{g}h$ is an analytic mapping of an open set of $\mathbf{R}^n$ into $\mathbf{C}^n$, which means that $\overset{-1}{g}h$ is given by n complex-valued analytic functions f_i $(i = 1, 2, ..., n)$. We choose complex extensions k_i of these functions and we take an open neighbourhood U of $\mathbf{x} = \overset{-1}{h}(y)$ in $\mathbf{C}^n$ and on which the functions k_i are all defined. The k_i define an analytic mapping k of U into $\mathbf{C}^n$, and I shall show that k is regular if U is taken small enough. This will be true if the k_i have non-zero Jacobian at $\mathbf{x}$, which is equivalent to saying that the derived mapping $k'_{\mathbf{x}}$ defined by differentiating the k_i at $\mathbf{x}$ maps the tangent vectors at $\mathbf{x}$ onto those at $\overset{-1}{g}(y)$. The restriction of $k'_{\mathbf{x}}$ to real vectors is the derivative of $\overset{-1}{g}h$ at $\mathbf{x}$ mapping the real vectors onto the space X of vectors tangential to $\overset{-1}{g}\{M \cap V(g)\}$ at $\overset{-1}{g}(y)$. The image of $k'_{\mathbf{x}}$ is the space spanned by X and $\sqrt{(-1)}X$ because of the linearity of $k'_{\mathbf{x}}$, so $k'_{\mathbf{x}}$ will be onto if $\sqrt{(-1)}X$ is transversal to X. The condition that M should be a real submanifold is exactly that X and $\sqrt{(-1)}X$ are transversal. It follows that k is regular, and gk is then a coordinate mapping of N which takes $U(gk) \cap \mathbf{R}^n$ onto $V(gk) \cap M$.

The atlas of $N-M$ combined with the mappings gk as y varies over

† I am indebted to Professor H. Cartan for drawing my attention to this theorem.

M give an atlas of N compatible with Λ_n^e. Its completion is the required atlas $\mathscr{B}$.

Ehresmann [(2) 417] states that, for any analytic manifold M, there is a neighbourhood N of the diagonal Δ in $M \times M$ which admits a complex analytic structure with Δ, isomorphic to M, as analytic real submanifold. This implies that a complex extension of any manifold exists as will be proved in § 2.† It is, however, an open question whether there is an isomorphism between the $2n$-dimensional analytic structures on N induced by $M \times M$ and the complex structure. An isomorphism certainly does exist when M admits an analytic locally-regular embedding in $\mathbf{R}^m$ ($m \geqslant \dim M = n$). In this case, a complex submanifold V of $\mathbf{C}^n$ can be defined by extending the $m-n$ analytic functions which define M locally, and V is a complex extension of M. Let g be the embedding of $M \times M$ in $\mathbf{C}^m$ given by first embedding $M \times M$ in $\mathbf{R}^m \times \mathbf{R}^m$ in the natural way, then by mapping $(\mathbf{x}, \mathbf{y})$ onto $\mathbf{x}+\sqrt{(-1)}(\mathbf{y}-\mathbf{x})$. Thus g is analytic, locally-regular, and takes Δ onto the real part of V. Moreover, the $2n$-dimensional tangent planes to $g(M \times M)$ and V coincide on the real part, so that the normal $(2m-2n)$-planes to V set up an analytic isomorphism between neighbourhoods of the real part in the two manifolds.

2. Existence theorem

Every real analytic manifold satisfying the second axiom of countability admits a complex extension.

2.1. Here we set up the basic structure required for the proof.

Choice of atlas. Assume that the real manifold M is covered by a family of coordinate neighbourhoods $V(f_i')$, indexed over the positive integers, such that each $V(f_i')$ is relatively compact and the covering is locally-finite of order $n+1$. The coordinate mappings f_i' form an atlas $\mathscr{A}'$. Such an atlas always exists and can, in fact, be constructed from a suitably fine simplicial decomposition of M. Next, refine $\mathscr{A}'$ to give an atlas $\mathscr{A}$ satisfying the same conditions as $\mathscr{A}'$, but such that $\overline{V(f_i)} \subset V(f_i')$. This can be done as follows. Consider $C_1 = \complement \bigcup_{i \neq 1} V(f_i')$, which is a closed subset of $V(f_1')$. Then, C_1 has an open neighbourhood V_1 whose closure is contained in $V(f_1')$, because C_1 and the frontier of $V(f_1')$ are non-intersecting closed sets and can be separated in the normal space $\overline{V(f_i')}$. This process is now repeated on $V(f_2')$ in the

† [*Note added in proof.*] The existence theorem can be proved by putting a complex analytic structure on N such that the $2n$-dimensional subordinate real structure is C^∞ isomorphic to the induced real structure on N.

covering $\{V_1, V(f'_2), V(f'_3),\ldots\}$, and so on, giving eventually $\{V_1, V_2, V_3,\ldots\}$. It is easily shown that $\{V_i\}$ forms a covering of M and that $\bar{V}_i \subset V(f'_i)$. The new atlas $\mathscr{A}$ is given by letting f_i be the restriction of f'_i to $\overset{-1}{f'_i}(V_i)$.

Complex extensions of coordinate changes. Let $r : \mathbf{C}^n \to \mathbf{R}^n$ be the retraction which maps complex vectors onto their real parts. For each pair i, j such that $i > j$, choose complex extensions ϕ'_{ij} and ϕ_{ij} of the changes of coordinates $\overset{-1}{f'_i}f'_j$ and $\overset{-1}{f_i}f_j$ respectively; Proposition 3 ensures that this is possible. We make the choice of the ϕ_{ij} in such a way that

$$\overline{U(\phi_{ij})} \subset U(\phi'_{ij}),$$

$$\Phi\begin{cases} r\{U(\phi_{ij})\} = U(\overset{-1}{f_i}f_j), \\ r\{V(\phi_{ij})\} = V(\overset{-1}{f_i}f_j). \end{cases}$$

The first condition can be satisfied because the closure of $U(\overset{-1}{f_i}f_j)$ is in $U(\phi'_{ij})$, and the second is possible because the inverse image of an open set of $\mathbf{R}^n$ under r is an open neighbourhood of the set. Let $\phi_{ij} = \overset{-1}{\phi}_{ji}$ for all $i < j$ and let ϕ_{ii} be the identity mapping of the set $U_i = \overset{-1}{r}\{U(f_i)\}$; similarly define the mappings ϕ'_{ij} for all pairs i, j.

2.2. *Method of proof.* It is known that $M = \Sigma\, U(f_i)/R$, where R is the equivalence relation

$$\mathbf{x}_i \equiv \mathbf{x}_j \quad \text{if } \mathbf{x}_i = \overset{-1}{f_i}f_j(\mathbf{x}_j).$$

By analogy, consider the relation S defined in $\Sigma\, U_i$ by

$$\mathbf{z}_i S \mathbf{z}_j \quad \text{if } \mathbf{z}_i = \phi_{ij}(\mathbf{z}_j).$$

The relation S is reflexive because $\mathbf{z}_i = \phi_{ii}(\mathbf{z}_i)$ and symmetric because $\phi_{ij} = \overset{-1}{\phi}_{ji}$. However, S may not necessarily be transitive. The procedure is therefore as follows:

Part 1. Replace the mappings ϕ_{ij} by suitable restrictions ϕ^*_{ij} making S^* an equivalence relation. The factor space is then like a complex extension except that it may not be Hausdorff.

Part 2. Replace the sets U_i by suitable subsets U^*_i such that $N = \Sigma\, U^*_i/S^*$ is Hausdorff and is a complex extension of M.

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Carriers $A_{\alpha q}$ *in* M. Consider sets of distinct positive integers

$$\alpha = \{i_1, i_2,..., i_q\}$$

such that
$$A_{\alpha q} = \bigcap_{i \in \alpha} V(f_i) \neq \emptyset:$$
that is, α indexes a non-empty carrier. It will be noted that
$$1 \leqslant q \leqslant n+1.$$
An intersection relation satisfied by the carriers is
$$A_{\gamma(q+1)} = A_{\alpha q} \cap A_{\beta q} \quad \text{if } \alpha, \beta \subset \gamma,\ \alpha \neq \beta.$$

Neighbourhoods $P_{\alpha\beta}$ *in* M. Associate with each set $A_{\beta(q+1)}$ the set $B_{\beta(q+1)}$ which is the union of all sets $A_{\alpha q}$ such that $\alpha \subset \beta$. Denote the frontier of an open set A by ∂A ($\partial A = \bar{A} - A$) and consider $\partial A_{\beta(q+1)} \cap B_{\beta(q+1)}$. The set $A_{\alpha q} - A_{\beta(q+1)}$ is closed in $B_{\beta(q+1)}$ if $\beta \supset \alpha$ because it is the complement of the union of all other carriers of order q contained in $B_{\beta(q+1)}$. It follows that $\partial A_{\beta(q+1)} \cap A_{\alpha q}$ is closed in $B_{\beta(q+1)}$, so the subset $\partial A_{\beta(q+1)} \cap B_{\beta(q+1)}$ of $B_{\beta(q+1)}$ is partitioned into $q+1$ closed sets of the form $\partial A_{\beta(q+1)} \cap A_{\alpha q}$. The space $B_{\beta(q+1)}$ is normal and therefore there exist $q+1$ non-intersecting open neighbourhoods $P^*_{\alpha\beta}$, respectively, of the $q+1$ closed sets. The set $A'_{\beta(q+1)}$, which is deduced from $\mathscr{A}'$ in the way that $A_{\beta(q+1)}$ was deduced from $\mathscr{A}$, is an open neighbourhood of $\bar{A}_{\beta(q+1)}$. Let $P_{\alpha\beta}$ be the intersection of $P^*_{\alpha\beta}$ with $A_{\alpha q}$ and with this neighbourhood.

The properties of the open sets $P_{\alpha\beta}$ can be summarized as

(i) $P_{\alpha\gamma} \cap P_{\beta\gamma} = \emptyset$ if $\alpha, \beta \subset \gamma$, $\alpha \neq \beta$,

(ii) $P_{\alpha\beta} \subset A'_{\beta(q+1)} \subset V(f'_i)$ if $i \in \beta$,

(iii) $P_{\alpha\beta}$ is a neighbourhood of $\partial A_{\beta(q+1)} \cap A_{\alpha q}$ in $A_{\alpha q}$.

Sets $A^i_{\alpha q}$, $P^i_{\alpha\beta}$ *in* $\mathbf{R}^n$. For each i in α, let $A^i_{\alpha q} = \overset{-1}{f_i}(A_{\alpha q})$. Thus we have

($\mathrm{A_I}$) $A^i_{\alpha q} = \overset{-1}{f_i} f_j(A^j_{\alpha q})$ if $i, j \in \alpha$,

($\mathrm{A_{II}}$) $A^i_{\alpha q} \cap A^i_{\beta q} = A^i_{\gamma(q+1)}$ if $i \in \alpha$, $\beta \subset \gamma$, $\alpha \neq \beta$.

Also, the closure of $A_{\alpha q}$ is in $A'_{\alpha q}$, and so the frontier points of $A^j_{\alpha q}$ will be mapped onto those of $A^i_{\alpha q}$ by ϕ'_{ij}. Let $P^i_{\alpha\beta} = \overset{-1}{f'_i}(P_{\alpha\beta})$ for each $i \in \beta$, giving

($\mathrm{P_I}$) $P^i_{\alpha\beta} = \overset{-1}{f'_i} f'_j(P^j_{\alpha\beta})$ if $i, j \in \beta$,

($\mathrm{P_{II}}$) $P^i_{\alpha\gamma} \cap P^i_{\beta\gamma} = \emptyset$ if $\alpha, \beta \subset \gamma$, $\alpha \neq \beta$,

($\mathrm{P_{III}}$) If $i \in \alpha$, $P^i_{\alpha\beta}$ is a neighbourhood of $\partial A^i_{\beta(q+1)} \cap A^i_{\alpha q}$ in $A^i_{\alpha q}$.

Sets $C^i_{\alpha q}$ *in* $\mathbf{C}^n$. I shall define open neighbourhoods $C^i_{\alpha q}$ of $A^i_{\alpha q}$ in U_i such that, for each pair i, j in α,
$$C^i_{\alpha q} = \phi_{ij}(C^j_{\alpha q}).$$

If we consider mappings such as

$$\theta_{ij} = \phi_{ik}\phi_{kl}\cdots\phi_{mj}$$

composed of mappings, where the pairs i, k; k, l; ...; m, j are taken from α, we note that each θ_{ij}, and in particular ϕ_{ij}, is a complex extension of the mapping $\overset{-1}{f_i}f_j$. We therefore choose the sets $C^i_{\alpha q}$ connected to $\mathbf{R}^n$ ensuring that the restrictions of θ_{ij} and ϕ_{ij} to $C^j_{\alpha q}$ are the same mapping for all i, j, θ by Proposition 2.

Let $\bar{\theta}_{ij}$ be a mapping similar to θ_{ij} above but let the bar indicate that every ordered pair l, m of α is included in the mappings ϕ_{lm} composed. Define $C^i_{\alpha q}$ to be the union of those connected components of $U(\bar{\theta}_{ij})$ which contain connected components of $A^i_{\alpha q}$.

It must first be shown that, if $C^{*j}_{\alpha q}$ is defined by another such mapping $\bar{\theta}^*_{kj}$, then $C^{*j}_{\alpha q} = C^j_{\alpha q}$. Assume, alternatively, that one of them, say $C^j_{\alpha q}$, has some points not included in the other. Let ϕ_{lm} be the first of the mappings in $\bar{\theta}^*_{kj}$ which does not map the full image of $C^j_{\alpha q}$ under previous components. Then, ϕ_{lm} also occurs somewhere in $\bar{\theta}_{ij}$ and, in this case, it does map the full image under previous components. The previous components in each case are complex extensions of $\overset{-1}{f_m}f_j$ so that they map $C^j_{\alpha q}$ in the same way. This leads to a contradiction from which we conclude that $C^{*j}_{\alpha q} = C^j_{\alpha q}$.

To prove that $C^i_{\alpha q} = \theta_{ij}(C^j_{\alpha q})$, let θ^*_{ji} include all ordered pairs from α which do not occur in θ_{ij}, so that $\theta_{ij}\theta^*_{ji}\theta_{ij}$ and $\theta_{ij}\theta^*_{ji}$ contain all ordered pairs. Hence, $\theta_{ij}\theta^*_{ji}\theta_{ij}$ maps $C^j_{\alpha q}$ and, consequently, $\theta_{ij}\theta^*_{ji}$ maps $\theta_{ij}(C^j_{\alpha q})$ which is therefore contained in $C^i_{\alpha q}$. The equality follows from a similar argument with $\overset{-1}{\theta_{ij}}$.

Complex carriers $D^i_{\alpha q}$. The sets $C^i_{\alpha q}$ are, to some extent, complex analogues of the $A^i_{\alpha q}$ but they do not satisfy the intersection relation $(\mathrm{A_{II}})$. The next step is to define subsets $D^i_{\alpha q}$ of $C^i_{\alpha q}$ such that

$(\mathrm{D_I})$ $D^k_{\alpha q} = \phi_{kj}(D^j_{\alpha q})$ if $k, j \in \alpha$,

$(\mathrm{D_{II}})$ $D^i_{\alpha q} \cap D^i_{\beta q} = D^i_{\gamma(q+1)}$ if $i \in \alpha$, $\beta \subset \gamma$, $\alpha \neq \beta$,

$(\mathrm{D_{III}})$ $D^i_{\alpha q}$ is an open neighbourhood of $A^i_{\alpha q}$ in $\overset{-1}{r}(A^i_{\alpha q})$.

These sets are defined by an inductive process starting with $D^i_{\alpha(n+1)} = C^i_{\alpha(n+1)}$ and defining $D^i_{\alpha q}$ in terms of $D^i_{\gamma(q+1)}$. Let $D^i_{\alpha\gamma}$ be the subset of $C^i_{\alpha q}$ given by

(*a*) $\mathbf{z} \in D^i_{\gamma(q+1)}$,

or (*b*) $r(\mathbf{z}) \in P^i_{\alpha\gamma}$,

or (*c*) $r(\mathbf{z}) \notin A^i_{\gamma(q+1)}$,

and let
$$D^i_{\alpha q} = \bigcap_{j,\gamma} \phi_{ij}(D^j_{\alpha\gamma}),$$
where the intersection ranges over all j in α and over all γ for which $P^i_{\alpha\gamma}$ is defined. It will be noted that the intersection contains a finite number of terms, because a given coordinate neighbourhood intersects only a finite number of others.

Assume ($\mathrm{D_{III}}$) true for each $D^i_{\gamma(q+1)}$. Then
$$\begin{aligned} A^i_{\alpha q} &\subset D^i_{\alpha\gamma}, \quad \text{by } (a), (b), (c),\\ &\subset C^i_{\alpha q}, \quad \text{by definition,}\\ &\subset \bigcap_{j\in\alpha} U(\phi_{ji}), \quad \text{by definition of } C^i_{\alpha q} \text{ in § 2.3,}\\ &\subset \overset{-1}{r}(A^i_{\alpha q}), \quad \text{by } (\Phi) \text{ for } \phi_{ij} \text{ in § 2.1.}\end{aligned}$$

Hence, ($\mathrm{A_I}$) gives $A^i_{\alpha q} \subset D^i_{\alpha q} \subset \overset{-1}{r}(A^i_{\alpha q})$. The subsets of $C^i_{\alpha q}$ given by (a) and (b) are open, so that the only possible non-interior points of $D^i_{\alpha\gamma}$ are in $\overset{-1}{r}(\partial A^i_{\gamma(q+1)})$, which is contained in (b). Hence, each $D^i_{\alpha\gamma}$ is open, which implies that $D^i_{\alpha q}$ is open. The sets $D^i_{\alpha(n+1)}$ certainly satisfy ($\mathrm{D_{III}}$); so ($\mathrm{D_{III}}$) is true for all q by induction.

Condition ($\mathrm{D_I}$) is satisfied because
$$\begin{aligned} D^k_{\alpha q} &= \bigcap_{i\in\alpha} \phi_{ki}\Big(\bigcap_{\gamma} D^i_{\alpha\gamma}\Big)\\ &= \bigcap_{i\in\alpha} \phi_{kj}\phi_{ji}\Big(\bigcap_{\gamma} D^i_{\alpha\gamma}\Big), \quad \text{since } D^i_{\alpha\gamma} \subset C^i_{\alpha q},\\ &= \phi_{kj}\Big(\bigcap_{i,\gamma} \phi_{ji}(D^i_{\alpha\gamma})\Big)\\ &= \phi_{kj}(D^j_{\alpha q}).\end{aligned}$$

We now verify condition $\mathrm{D_{II}}$. Remember that $D^i_{\alpha\gamma} \subset r(A^i_{\alpha q})$. Hence, if $i \in \alpha$, $\beta \subset \gamma$, it follows from ($\mathrm{A_{II}}$) that
$$D^i_{\alpha\gamma} \cap D^i_{\beta\gamma} \subset \overset{-1}{r}(A^i_{\gamma(q+1)}).$$
Also, by (a), (b), (c), any point of intersection of the two sets is either in $D^i_{\gamma(q+1)}$, or in the intersection of two sets $\overset{-1}{r}(P^i_{\alpha\gamma})$ and $\overset{-1}{r}(P^i_{\beta\gamma})$. Hence ($\mathrm{P_{II}}$) shows that
$$D^i_{\gamma(q+1)} = D^i_{\alpha\gamma} \cap D^i_{\beta\gamma}.$$
Condition ($\mathrm{D_{II}}$) follows by using
$$D^i_{\gamma(q+1)} = \phi_{ij}(D^j_{\gamma(q+1)}) \quad \text{and} \quad D^i_{\alpha q} \subset D^i_{\alpha\gamma}.$$

2.4. We are now able to proceed with the proof as indicated in § 2.2.

Part 1. Let ϕ^*_{ij} be the restriction of ϕ_{ij} to $D^j_{(ij)2}$ and let $\phi^*_{ii} = \phi_{ii}$. Thus, $\overset{-1}{\phi^*_{ij}}$ is the restriction of ϕ_{ji} to

$$\phi^*_{ij}(D^j_{(ij)2}) = D^i_{(ij)2},$$

which is ϕ^*_{ji}. Also, if $\mathbf{z}_i = \phi^*_{ij}(\mathbf{z}_j)$ and $\mathbf{z}_j = \phi^*_{jk}(\mathbf{z}_k)$, then

$$\begin{aligned} \mathbf{z}_j &\in D^j_{(ij)2} \bigcap \phi^*_{jk}(D^k_{(ij)2}) \\ &= D^i_{(ij)2} \bigcap D^j_{(jk)2}, \quad \text{by } (\mathrm{D_I}), \\ &= D^j_{(ijk)3}, \quad \text{by } (\mathrm{D_{II}}). \end{aligned}$$

Hence, $\mathbf{z}_k \in D^k_{(ijk)3}$ by $(\mathrm{D_I})$, and $\mathbf{z}_i = \phi^*_{ik}(\mathbf{z}_k)$ by the uniqueness property of θ_{ik} mappings on $C^k_{\alpha q}$.

The relation S^*,

$$\mathbf{z}_i\, S^* \mathbf{z}_j \quad \text{if } \mathbf{z}_i = \phi^*_{ij}(\mathbf{z}_j),$$

is an equivalence relation in $\Sigma\, U_i$.

Part 2. The quotient mapping of $\Sigma\, U_i/S^*$ is open and this makes it easy to specify exactly which pairs of points of $\Sigma\, U_i$ give rise to separable points of the quotient. If two points are in the same U_i, they clearly have separable images. Let $\mathbf{z}_i$ and $\mathbf{z}_j$ be points of U_i and U_j respectively whose images are not separable. Then $\mathbf{z}_i$ cannot be in $U(\phi^*_{ji})$, for it would then be equivalent to a point of U_j. On the other hand, any neighbourhood of $\mathbf{z}_i$ must have points equivalent to points in U_j, and so $\mathbf{z}_i$ must be a frontier point of $U(\phi^*_{ji})$. Similarly, $\mathbf{z}_j$ is a frontier point of $U(\phi^*_{ij})$. If $\mathbf{z}_i \neq \phi'_{ij}(\mathbf{z}_j)$, we can separate these points by enclosing them in open sets W_i and W'_j, respectively, of U_i, and the quotient images of the open sets W_i and $\phi'_{ji}(W'_j \cap U(\phi'_{ji}))$ are non-intersecting, which contradicts the hypothesis that $\mathbf{z}_i$ and $\mathbf{z}_j$ have non-separable images. We conclude that the only pairs of points of $\Sigma\, U_i/S^*$ which are not Hausdorff-separable are of the form $\mathbf{z}_j$, $\phi'_{ij}(\mathbf{z}_j)$, where $\mathbf{z}_j$ is a frontier point of $D^j_{(ij)2}$. Let $C'^i_{\alpha q}$ be obtained from the ϕ'_{ij} in the way that $C^i_{\alpha q}$ are obtained from the ϕ_{ij}. Let

$$Q^i_j = \overset{-1}{r}(P^i_{(i)(ij)}) \bigcap C'^i_{(ij)2} \bigcap \phi'_{ij}\left(\overset{-1}{r}(P^j_{(j)(ij)}) \cap C'^j_{(ij)2}\right).$$

Thus Q^i_j is an open neighbourhood of $\partial A^i_{(ij)2}$ in U_i and

$$Q^i_j \bigcap \phi'_{ij}(Q^j_i) = \varnothing, \quad \text{by } (\mathrm{P_I}) \text{ and } (\mathrm{P_{II}}).$$

Let $U_{ij} = (U_i - \partial D^i_{(ij)2}) \bigcup Q^i_j$, so that U_{ij} is a neighbourhood of $U(f_i)$ in U_i. It is then clear that $U_{ij} + U_{ji}$, factored by the restriction of S^* to it, is Hausdorff. Hence let

$$U^*_i = \bigcap_j U_{ij},$$

where the intersection is taken over the j for which

$$V(f_i) \cap V(f_j) \neq \emptyset.$$

The intersection has a finite number of terms so that U_i^* is open and the space $\Sigma\, U_i^*/S^*$ being Hausdorff is the required complex extension of M.

3. Extension of analytic structures

It frequently occurs that analytic structures on the real manifold M are of types which are uniquely determined by giving analytic functions h_a^i $(a = 1, 2, \ldots, r)$ on each open set $U(f_i)$ of the atlas $\mathscr{A}$ used in the construction of the complex extension. Usually, the structure and $\mathscr{A}$ do not determine the functions uniquely: the h_a^i may represent an analytic equivalence class of sets of r functions defined on $U(f_i)$. Also the h_a^i functions do not define a structure unless certain analytic identities are satisfied in the coordinate overlaps. These identities involve the change of coordinates $\overset{-1}{f}_i f_j$, the functions h_a^j, the functions $h_a^i \overset{-1}{f}_i f_j$, and their derivatives. An extension of such a structure is given by constructing an extension N of the manifold M as in § 2 but,

(i) choosing the sets U_i in § 2.1 as the subsets of $\overset{-1}{r}(U(f_i))$ on which complex extensions k_a^i of the h_a^i are defined, and,

(ii) choosing the ϕ_{ij} (defining them over sufficiently small neighbourhoods) in such a way that the complex extensions of the analytic identities are satisfied.

Suppose, for example, that M is a Riemannian space, so that the structure is determined by giving the components g_{pq} of the metric tensor in each coordinate neighbourhood and the identities in the overlaps are the usual tensor law of transformation. The complex extension is a complex manifold N with a complex tensor g_{pq} which, having a rank dim M, does not define a metric on N but which is such that the imaginary complex directions at a point of M are realized as directions transversal to M in N. For instance, the null cone of a positive definite metric appears in the complex extension.

An analytic almost-complex structure on a $2n$-dimensional manifold M is determined by giving an analytic field of complex n-elements X_n such that X_n and $\bar{X}_n$ have only one point in common, the origin of the tangent space [(**2**) 414]. The almost-complex structure on M can be extended to a complex manifold N as above, and the field of imaginary n-elements X_n give a field of complex n-elements Y_n in N which are transversal to M.

The almost-complex structure on M is complex if and only if there is a subatlas of preferred coordinates such that X_n is given by

$$du^i+\surd(-1)du^{i'} = 0 \quad (i = 1,2,\ldots,n;\ i' = i+n).$$

A necessary and sufficient condition that the almost-complex structure on M should be complex is that the field of complex n-elements Y_n is completely integrable in N. The condition is obviously necessary because the equations $du^i+\surd(-1)du^{i'} = 0$ are completely integrable. Conversely, if Y_n are tangent elements to laminas given locally by putting n analytic functions g^i equal to constants, then

$$g^i = h^i+\surd(-1)h^{i'}$$

on M, where h^i, $h^{i'}$ are real analytic functions. These $2n$ functions are independent because the laminas are transversal, and thus they give the preferred coordinate systems. The integrability condition is the usual one when expressed in terms of the forms, that is,

$$d\omega^i \equiv 0 \ (\mathrm{mod}\,\omega^i),$$

where ω^i are the complex forms defining X_n.

Patterson's theorem [(4) 266], that an analytic almost-Kähler metric is always Kähler, can be deduced in N from the fact that parallel planes are integrable.

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ON A FUNCTIONAL EQUATION

By T. W. CHAUNDY *and* J. B. McLEOD (*Oxford*)

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1. It is a problem of some interest in the statistical thermodynamics of mixtures† to obtain the general solution of the functional equation

$$f(x)+uf(vx) = Uf(Vx), \tag{1}$$

where x, u, v are independent parameters, $f(x)$ is the unknown function, required to be continuous, and U, V are (unknown) functions of u, v alone whose form will depend (presumably) on the form of f. For the purposes of the problem we need consider only positive values of x, u, v, U, V; but for the mathematical analysis it is reasonable, in view of the symmetry of (1), to admit values of u, U of either sign, and this may lead to imaginary values of v, V, as may be seen from the solutions asserted below.

Since V is defined implicitly, it may well be many-valued: to distinguish between these possible branches it is not unreasonable to impose on $V(u,v)$ the condition of being continuous.

We show that the general solution of (1) is

$$f(x) = Ax^a+Bx^b, \tag{2}$$

where A, B, a, b are constants; U, V are then given by

$$1+uv^a = UV^a, \qquad 1+uv^b = UV^b. \tag{3}$$

There are two exceptional or limiting cases of this. Corresponding to the limit $b \to a$, we may have

$$f(x) = (A+B\log x)x^a, \tag{4}$$

where now $\quad 1+uv^a = UV^a, \qquad u\log v = U\log V. \quad (5)$

Corresponding to $B = 0$, we may have

$$f(x) = Ax^a, \tag{6}$$

and then u, v, U, V are connected by the single relation

$$1+uv^a = UV^a. \tag{7}$$

We exclude as sufficiently pointless the solution in which $A = 0$, $B = 0$ and $f(x)$ is identically zero.

† See W. B. Brown, *Phil. Trans.* (A) 250 (1957) 175.

Quart. J. Math. Oxford (2), 9 (1958), 202-6.

2. Suppose first that we can deduce from (1) a *two-term* relation

$$f(xt) = Tf(x), \tag{8}$$

where x, t are independent parameters and T is a function of t alone. Taking h as a particular value of x we have

$$f(h)f(xt) = f(ht)f(x).$$

The solution of this is well known; it is, perhaps, even better known if we write it as

$$\log f(e^k)+\log f(e^{u+v}) = \log f(e^{v+k})+\log f(e^u),$$

i.e.
$$F(u+v)+F(k) = F(u)+F(v+k), \tag{9}$$

where u, v are independent. It is notorious that the only *continuous* solution of (9) is

$$F(u) = au+c,$$

where a, c are constants, and so the only continuous solution of (8) is

$$f(x) = Ax^a,$$

where $A = e^c$. This is the solution (6).

3. Returning to (1) in the general case take t any value of x and, keeping t a disposable parameter, write

$$u = -f(t)/f(vt),$$

so that U, V are now functions of the independent parameters v, t: say $U = U(t,v)$, $V = V(t,v)$. Thus

$$f(x)-f(t)f(vx)/f(vt) = U(t,v)f\{xV(t,v)\}. \tag{10}$$

This gives, when $x = t$, either

$$\text{(i) } U(t,v) = 0 \quad \text{or} \quad \text{(ii) } f\{tV(t,v)\} = 0.$$

If (i) holds for any t, v, we have the two-term relation

$$f(x)f(vt) = f(t)f(vx)$$

in three independent parameters, and this is covered by § 2 and leads to the solution (6).

4. Alternatively, from (ii), we have, for all t, v,

$$f(x) = 0,$$

where
$$x \equiv x(t,v) = tV(t,v).$$

Then $x(t,v)$ is continuous by the continuity imposed on V, and so, in general, $f(x)$ vanishes for a continuous region of values of x: in other words it is identically zero over an interval. Rejecting this we have, as the only alternative, that $x(t,v)$ is some constant c: that is,

$$V(t,v) = ct^{-1}, \tag{11}$$

where c is a zero of $f(x)$.

We accordingly rewrite (10) as

$$f(x)-\frac{f(t)}{f(vt)}f(vx) = U(t,v)f(ct^{-1}x), \tag{12}$$

where $f(c) = 0$. Putting $x = c$ gives us

$$U = -\frac{f(t)f(cv)}{f(vt)f(c^2t^{-1})}.$$

Write further $t = cw^{-1}$ and we get

$$f(x)-\frac{f(cw^{-1})}{f(cvw^{-1})}f(vx) = -\frac{f(cw^{-1})f(cv)}{f(cvw^{-1})f(cw)}f(wx),$$

i.e.

$$\frac{f(cvw^{-1})}{f(cv)f(cw^{-1})}f(x) = \frac{f(vx)}{f(cv)}-\frac{f(wx)}{f(cw)}. \tag{13}$$

Since $f(x)$ is not to vanish identically, we have $f(x_0) \neq 0$ for some x_0. By adjusting constants in x, f we may sufficiently write this $f(1) = 1$. This will change the value of c, but we sufficiently continue to indicate it by the same letter. Then $x = 1$ in (13) gives

$$\frac{f(cvw^{-1})f(1)}{f(cv)f(cw^{-1})} = \frac{f(v)}{f(cv)}-\frac{f(w)}{f(cw)},$$

and so we can write (13) as

$$\frac{f(vx)-f(v)f(x)}{f(cv)} = \frac{f(wx)-f(w)f(x)}{f(cw)}.$$

Thus each side must be independent of v, w and accordingly equal to some function of x only. We write therefore

$$f(vx) = f(v)f(x)-f(cv)h(x). \tag{14}$$

Symmetry in x, v gives

$$f(cv)h(x) = h(v)f(cx),$$

and so $h(x) = Af(cx)$ for some constant A. Then (14) is

$$f(vx) = f(v)f(x)-Af(cv)f(cx),$$

and we write this more conveniently

$$f(vx) = f(v)f(x)-g(v)g(x), \tag{15}$$

where $g(x) = \sqrt{A}f(cx)$, so that $g(1) = 0$. Substituting vx, v^{-1} for x, v, we have

$$f(x) = f(v^{-1})f(vx)-g(v^{-1})g(vx).$$

With (15), eliminating $f(vx)$, we get

$$g(vx) = -\frac{1-f(v)f(v^{-1})}{g(v^{-1})}f(x)-\frac{f(v^{-1})g(v)}{g(v^{-1})}g(x). \tag{16}$$

Here $x = 1$ gives, since $g(1) = 0$,

$$f(v)f(v^{-1})-g(v)g(v^{-1}) = 1, \tag{17}$$

and (16) becomes

$$g(vx) = g(v)\left\{f(x)-\frac{f(v^{-1})}{g(v^{-1})}g(x)\right\}. \tag{18}$$

5. From (15), (18) we can write

$$f(vx)-\lambda g(vx) = g(v)\left[\left\{\frac{f(v)}{g(v)}-\lambda\right\}f(x)-\left\{1-\frac{\lambda f(v^{-1})}{g(v^{-1})}\right\}g(x)\right], \tag{19}$$

where λ is a disposable multiplier. The right-hand member is a multiple of $f(x)-\lambda g(x)$ if

$$\lambda^2-\lambda\left\{\frac{f(v)}{g(v)}+\frac{f(v^{-1})}{g(v^{-1})}\right\}+1 = 0. \tag{20}$$

We can reduce the middle term: for, from (18),

$$\frac{f(x)}{g(x)}-\frac{f(v^{-1})}{g(v^{-1})} = \frac{g(vx)}{g(v)g(x)} = \frac{f(v)}{g(v)}-\frac{f(x^{-1})}{g(x^{-1})}$$

by interchange of x, v. Then

$$\frac{f(v)}{g(v)}+\frac{f(v^{-1})}{g(v^{-1})} = \frac{f(x)}{g(x)}+\frac{f(x^{-1})}{g(x^{-1})} = 2h, \tag{21}$$

a constant, since it is independent of v, x.

Accordingly λ satisfies the equation

$$\lambda^2-2h\lambda+1 = 0 \tag{22}$$

and has, in general, two distinct constant values p and q $(= p^{-1})$ say. With these, (19) gives two equations of the form

$$f(vx)-pg(vx) = P(v)\{f(x)-pg(x)\},$$

$$f(vx)-qg(vx) = Q(v)\{f(x)-qg(x)\}.$$

By § 2 these give, when we remove the condition $f(1) = 1$,

$$f(x)-pg(x) = Rx^a, \qquad f(x)-qg(x) = Sx^b$$

for some constants R, S, a, b, and elimination of $g(x)$ gives the required form (2).

6. Exceptionally, (22) has equal roots when $h = \pm 1$. If we change the sign of g, we change the sign of h and of λ, but nothing essential alters. In this exceptional case we may therefore take $h = 1$ and so $\lambda = 1$. Then, with the help of (17), (19) becomes

$$f(vx)-g(vx) = \{f(v)-g(v)\}\{f(x)-g(x)\},$$

and so, still with $f(1) = 1$,

$$f(x)-g(x) = x^a \tag{23}$$

for some constant a. Write

$$g(x) = x^a G(x).$$

Then from (15), (23) we get

$$G(vx) = G(x) + G(v),$$

say

$$G(e^{w+y}) = G(e^y) + G(e^w),$$

of which the only continuous solution is

$$G(e^y) = Cy, \quad \text{i.e.} \quad G(x) = C \log x.$$

Thus, removing the condition $f(1) = 1$, we get finally

$$f(x) = (A + B \log x)x^a,$$

which is the limiting form (4).

ON THE COMMUTATOR SUBRING

By J. B. McLEOD (*Oxford*)

[Received 1 August 1957; in revised form 30 December 1957]

1. Introduction

LET S be a ring with a subfield K such that the elements of K commute with all elements of S. Let S have a unit element in K. S is said to be of *freedom* f over K if f is the minimum number of elements of S which, together with K, generate S polynomial-wise.

If $f = 1$, S is commutative. If $f = 2$ and K is of infinite characteristic, I prove that the commutator subring C of S (the subring of S generated by elements of the form $r \circ s = rs-sr$ with r, s in S) is a two-sided ideal of S.†

In § 5, we obtain examples of S

(i) with $f = 2$ and $K = GF(2)$;

(ii) with $f \geqslant 3$ and K arbitrary,

in which the commutator subring of S is *not* an ideal.

2. THEOREM. *Let $f = 2$ and K be of infinite characteristic. Then C is an ideal of S.*

Proof. Let x, y, together with K, generate S. If $r, s \in S$, we write $r \equiv s$ if $r-s \in C$. To prove the theorem, one easily sees that it is enough to prove that, for each r, s, t in S which are *monomials* in x, y,

$$r(s \circ t) \equiv 0, \qquad (s \circ t)r \equiv 0.$$

Hence it certainly suffices to prove the following lemma:

LEMMA. *Let m_1, n_1,..., m_r, n_r be* non-negative integers. Then

$$x^{m_1} y^{n_1} \dots x^{m_r} y^{n_r} \equiv x^a y^b,$$

where $$a = \sum_1^r m_i, \qquad b = \sum_1^r n_i.$$

The proof of this lemma is by induction. The result is trivial if $r = 1$. The proof for $r = 2$ is given in § 3. The general argument by induction from $r = n \geqslant 2$ to $r = n+1$ is given in § 4.

† The problem arose from a discussion with Dr. S. A. Jennings. Cf. the introduction to his paper, *Duke Math. J.* 9 (1942) 341–55.

Quart. J. Math. Oxford (2), 9 (1958), 207-9.

for some constant a. Write

$$g(x) = x^a G(x).$$

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3. Proof of the lemma for $r = 2$

If $a = 0$ or $a = 1$, the lemma is easily verified directly. Suppose that $a \geqslant 2$ and let m, n be arbitrary integers such that

$$a-1 \geqslant m \geqslant 1, \qquad b \geqslant n \geqslant 0.$$

Now

$$(x^{m-1}y^n x^{a-m} - x^m y^n x^{a-m-1})(xy^{b-n} - y^{b-n}x) \equiv 0,$$

and hence

$$x^{m-1}y^n x^{a-m+1}y^{b-n} - x^{m-1}y^n x^{a-m}y^{b-n}x - x^m y^n x^{a-m}y^{b-n} + x^m y^n x^{a-m-1}y^{b-n}x \equiv 0,$$

$$x^{m-1}y^n x^{a-m+1}y^{b-n} - x^m y^n x^{a-m}y^{b-n} - x^m y^n x^{a-m}y^{b-n} + x^{m+1}y^n x^{a-m-1}y^{b-n} \equiv 0,$$

$$x^{m-1}y^n x^{a-(m-1)}y^{b-n} - x^m y^n x^{a-m}y^{b-n} \equiv x^m y^n x^{a-m}y^{b-n} - x^{m+1}y^n x^{a-(m+1)}y^{b-n}. \tag{3.1}$$

Thus, putting $m = 1, 2,\ldots, a-1$ successively in (3.1), we have

$$\left.\begin{aligned} y^n x^a y^{b-n} - xy^n x^{a-1}y^{b-n} &\equiv xy^n x^{a-1}y^{b-n} - x^2y^n x^{a-2}y^{b-n} \\ &\equiv x^2y^n x^{a-2}y^{b-n} - x^3y^n x^{a-3}y^{b-n} \\ &\quad \cdots\cdots\cdots \\ &\equiv x^{a-2}y^n x^2 y^{b-n} - x^{a-1}y^n xy^{b-n} \\ &\equiv x^{a-1}y^n xy^{b-n} - x^a y^b \end{aligned}\right\}. \tag{3.2}$$

It follows that

$$\sum_{i=0}^{a-1} (x^i y^n x^{a-i}y^{b-n} - x^{i+1}y^n x^{a-i-1}y^{b-n}) \equiv a(x^{a-1}y^n xy^{b-n} - x^a y^b). \tag{3.3}$$

The left-hand member of (3.3) is

$$y^n x^a y^{b-n} - x^a y^b \in C.$$

Thus, from (3.3), it follows that, if K is of infinite characteristic, then

$$\begin{aligned} x^a y^b &\equiv x^{a-1}y^n xy^{b-n} \\ &\equiv x^{a-s}y^n x^s y^{b-n}, \end{aligned}$$

for each integer s such that $0 \leqslant s \leqslant a$, by (3.2). This proves the lemma for $r = 2$.

4. Proof of the lemma for $r > 2$

We now take $r > 2$ and, assuming the truth of the lemma for $r-1$, prove it for r. Then

$$(x^{m_1}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}}x^{m_{r-1}} - x^{m_1+m_{r-1}}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}})(y^{n_{r-1}}x^{m_r}y^{n_r} - y^{n_{r-1}+n_r}x^{m_r}) \equiv 0,$$

$$\begin{aligned} &x^{m_1}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}}x^{m_{r-1}}y^{n_{r-1}}x^{m_r}y^{n_r} - x^{m_1}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}}x^{m_{r-1}}y^{n_{r-1}+n_r}x^{m_r} - \\ &\quad - x^{m_1+m_{r-1}}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}+n_{r-1}}x^{m_r}y^{n_r} + x^{m_1+m_{r-1}}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}+n_{r-1}+n_r}x^{m_r} \equiv 0. \end{aligned} \tag{4.1}$$

The second term in (4.1) is congruent with

$$-x^{m_1+m_r}y^{n_1}\ldots x^{m_{r-2}}y^{n_{r-2}}x^{m_{r-1}}y^{n_{r-1}+n_r},$$

and then by the induction hypothesis the second and third terms are congruent with $-x^a y^b$ and the fourth term is congruent with $x^a y^b$. Hence the first term is congruent with $x^a y^b$, and the lemma is proved.

5. It remains to give examples of S

(i) with $f = 2$ and $K = GF(2)$;

(ii) with $f \geqslant 3$ and K arbitrary,

in which the commutator subring of S is not an ideal.

(i) We take our example from the free ring generated by non-commuting indeterminates x, y over $GF(2)$. Then

$$x^2y^2 - xyxy = x(xy-yx)y,$$

and so belongs to the ideal generated by the commutator subring C. But

$$x^2y^2 - xyxy \notin C.$$

For the only product of commutators which can be used to 'connect' x^2y^2 and $xyxy$ is $(xy-yx)^2$. But

$$\begin{aligned}(xy-yx)^2 &= xyxy - yx^2y - xy^2x + yxyx \\ &\equiv 2(xyxy - x^2y^2).\end{aligned}$$

(ii) We take our example from the free ring generated by f non-commuting indeterminates $x, y, z, \ldots$ over K. Then

$$x(yz-zy)$$

belongs to the ideal generated by C, but not to C. For, to express $x(yz-zy)$ as a member of C, we must use a commutator of the third degree, e.g.

$$(xy) \circ z.$$

But any such commutator connects two terms in which the cyclic order of the factors is the same, and this is not the case for

$$x(yz-zy).$$

ONE-DIMENSIONAL CHARACTERISTICS OF A PARTIAL DIFFERENTIAL EQUATION OF THE SECOND ORDER, WITH ANY NUMBER OF INDEPENDENT VARIABLES

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In any attempt to extend the classical theory of partial differential equations of the second order to equations with more than two independent variables, the first problem is that of defining 'characteristic multiplicities'. Goursat [(**2**) 219 chap. x] has pointed out that, out of several possible definitions, two are naturally indicated. Firstly, if there be $m+1$ independent variables, one can start from the problem of Cauchy, generalized, and define a characteristic multiplicity of order n $(n \geqslant 2)$ to be an m-dimensional multiplicity of elements of contact of order n, contained in an infinity of integral multiplicities. This definition, adopted by J. Beudon (**1**), is satisfactory up to a point, but does not lead to an extension of Darboux's method.

The second definition is as follows. Consider an equation of the second order, $\phi = 0$ say, with one dependent and $m+1$ independent variables. Then we define a *characteristic multiplicity* of order n $(n \geqslant 2)$ to be a *one*-dimensional multiplicity of elements of contact of order n, contained in at least one integral multiplicity, and satisfying at least one total differential equation distinct from the equation $d\phi = 0$ and the equations of contact, which contains the differential of at least one derivative of order n, and which is independent of the integral containing the multiplicity. This definition is analogous to that adopted by Natani (**3**) for characteristics of the first order.

I have shown (**4**) that, when there are three independent variables, an equation of the second order admits two families of characteristics of this kind, if it be of rank 2, one family if it be of rank 1, but none if it be of rank 3. I have also (**5**) extended the definition of rank to equations with any number of independent variables and shown that the rank is invariant under contact transformation.

I shall now extend this result to equations with $m+1$ independent variables; and we shall see that such equations admit characteristics of this kind if the rank be 2 or 1, but not if it be 3 or more. To simplify

the writing of the proof, I shall consider characteristics of the second order only. The definition of characteristics of higher order follows exactly the same lines, and the results are entirely similar, the only difficulty being the greater complication of the notations required.

Let the independent variables be x, $y_1,..., y_m$, and let z be the dependent variable. Let

$$\partial z/\partial x = p, \quad \partial z/\partial y_i = q_i, \quad \partial^2 z/\partial x^2 = r, \quad \partial^2 z/\partial x \partial y_i = s_i,$$
$$\partial^2 z/\partial y_i \partial y_j = t_{ij} \quad (i,j = 1,..., m).$$

Also let

$$\partial^3 z/\partial x^2 \partial y_i = r_i, \quad \partial^3 z/\partial x \partial y_i \partial y_j = s_{ij}, \quad \partial^3 z/\partial y_i \partial y_j \partial y_k = t_{ijk}$$
$$(i,j,k = 1,..., m).$$

We may always suppose that the given equation contains the derivative r. For, if it does not contain $\partial^2 z/\partial x^2$ but contains any one of $\partial^2 z/\partial y_i^2$ $(i = 1,..., m)$, a change of notation reduces it to the required form. Again, if the equation does not contain $\partial^2 z/\partial x^2$ nor any of $\partial^2 z/\partial y_i^2$ $(i = 1,..., m)$, but contains, say, $\partial^2 z/\partial y_1 \partial y_2$, the change of variables

$$y_1+y_2 = X, \quad y_1-y_2 = Y_1, \quad x = Y_2, \quad y_i = Y_i \quad (i \geqslant 3)$$

gives an equation containing $\partial^2 z/\partial X^2$.

Thus we suppose that the given equation contains the derivative r. We shall deal with analytic equations only, and consequently we may suppose the given equation solved for r. Let the given equation be

$$r+F\begin{Bmatrix} x, y_1,..., y_m;\ z;\ p, q_1,..., q_m;\ s_1,..., s_m; \\ t_{11}, t_{12},..., t_{mm} \end{Bmatrix} = 0, \qquad (1)$$

where F is an analytic function of its arguments in the neighbourhood of a set of initial values.

We may suppose that every partial derivative involving two or more differentiations with respect to x is expressed, in terms of the variables and the remaining partial derivatives, by means of (1) and the equations derived from (1) by differentiation. Thus the only equations of contact which we need consider are those containing partial derivatives which involve not more than one differentiation with respect to x. Let us now use the symbol dF/dy_i to indicate the result of differentiating F with respect to y_i, treating z and each partial derivative as a function of x, $y_1,..., y_m$; and let (dF/dy_i) indicate that in calculating dF/dy_i we omit all terms involving derivatives of z of the third order. Let

$$\partial F/\partial s_i = S_i, \qquad \partial F/\partial t_{ij} = T_{ij} \quad (i,j = 1,..., m).$$

Then, with this notation, differentiating (1) with respect to y_i, we have

$$r_i = -(dF/dy_i) - \sum_{j=1}^{m} S_j s_{ij} - \sum_{j=1}^{m} T_{jj} t_{ijj} - \sum_{j<k}\sum T_{jk} t_{ijk} \quad (i = 1,\dots,m). \tag{2}$$

The equations of contact of the third order, when we use (1) to express the value of r, are

$$\left.\begin{aligned} dz - p\,dx - \sum_{j=1}^{m} q_j\,dy_j &= 0 \\ dp + F\,dx - \sum_{j=1}^{m} s_j\,dy_j &= 0 \\ dq_i - s_i\,dx - \sum_{j=1}^{m} t_{ij}\,dy_j &= 0 \quad (i = 1,\dots,m) \end{aligned}\right\}, \tag{3}$$

$$ds_i = r_i\,dx + \sum_{j=1}^{m} s_{ij}\,dy_j \quad (i = 1,\dots,m), \tag{4}$$

$$dt_{ij} = s_{ij}\,dx + \sum_{k=1}^{m} t_{ijk}\,dy_k \quad (i \leqslant j;\ i,j = 1,\dots,m). \tag{5}$$

Substituting for r_i from (2) in (4) and bearing in mind that $s_{ij} = s_{ji}$, etc., we have

$$ds_i + (dF/dy_i)dx = \sum_{j=1}^{m} (dy_j - S_j\,dx)s_{ij} - \sum_{j=1}^{m} T_{jj} t_{ijj}\,dx - \sum_{j<k}\sum T_{jk} t_{ijk}\,dx$$
$$(i = 1,\dots,m). \tag{6}$$

Now, given any integral element of contact of the second order, i.e. given any set of values of the variables and the partial derivatives of the first and second orders, satisfying (1), it follows from the usual existence theorem that an infinity of integrals of (1) exist admitting this element, and such that all those partial derivatives of the third order which occur on the right of (5) and (6) take, simultaneously, any arbitrarily chosen set of values. Thus suppose, if possible, that a multiplicity of integral elements of contact of the second order satisfies a total differential equation, distinct from $dr + dF = 0$ and equations (3), containing at least one of ds_i, dt_{ij} $(i,j = 1,\dots,m)$, which is independent of the integral containing the multiplicity. Then the right-hand sides of the equations (5) and (6), considered as linear forms in the partial derivatives of the third order, must be linearly dependent for the values of dy_i/dx $(i = 1,\dots,m)$ associated with the multiplicity. For, if this were not so, we could regard dy_j $(j = 1,\dots,m)$, dx as constants and solve (5) and (6) for certain of the partial derivatives of the third order; and, this being done, we could certainly find integrals of (1) such that, for the given values of the dy_i/dx $(i = 1,\dots,m)$ and the given element of

contact of the second order, ds_i, dt_{ij} $(i,j = 1,\dots,m)$ assume any arbitrarily chosen set of values. Thus these differentials could not be restricted to satisfying any total differential equation independent of the integral of (1).

We therefore require to investigate the condition that the right-hand sides of (5) and (6) are not linearly independent forms in the s_{ij}, t_{ijk} $(i,j,k = 1,\dots,m)$.

Noticing that $s_{ij} = s_{ji}$, $t_{ijk} = t_{ikj}$, etc., we require to arrange the distinct derivatives of the third order in a definite order. We write the pairs of numbers i, j in the sequence

$$(1,1),\ (1,2),\dots,\ (1,m);\ (2,2),\ (2,3),\dots,\ (2,m);\ \dots;\ (m,m):$$

and the triplets (i,j,k) in the sequence

$$(1,1,1),\ (1,1,2),\dots,\ (1,1,m);\ (1,2,2),\ (1,2,3),\dots,\ (1,2,m);\ \dots;$$
$$(2,2,2),\dots,\ (2,2,m);\ \dots;\ (m,m,m).$$

We now require three lemmas

(A) In the first sequence, the m pairs containing a selected number i occur in ascending order of the second number. This is obvious if $i = 1$. If $i > 1$, we simply observe that these pairs occur in the order $(1,i),\dots,$ $(i-1,i)$, (i,i), $(i,i+1),\dots,$ (i,m).

(B) In the second sequence, the triplets containing any selected pair of numbers j and k $(j \leqslant k)$ occur in ascending order of the remaining number. For there are three possibilities: $i < j \leqslant k$ (occurring only if $j > 1$), $j \leqslant r < k$ (occurring only if $j < k$), and $j \leqslant k \leqslant s$. But (i,j,k) precedes (j,r,k) which precedes (j,k,s), since $i < j$ and $r < k$; and the result follows immediately.

(C) In the sequence of triplets, those containing one selected number i occur in the order of the sequence of pairs, applied to the other two numbers. The possibilities are $r \leqslant s < i$, $u < i \leqslant v$ (these occurring only if $i > 1$), and $i \leqslant j \leqslant k$. We require to show that the triplets (r,s,i), (u,i,v), (i,j,k) occur in the same order as the pairs (r,s), (u,v), (j,k). Clearly the set of triplets (r,s,i) occur in the same order as that in which the pairs (r,s) occur in the sequence of pairs; and the same is true of the (u,i,v) and (u,v), and of the (i,j,k) and (j,k). If $r < u$, (r,s,i) occurs before (u,i,v), and (r,s) before (u,v); and vice versa if $r > u$. If $r = u$, (u,s,i) occurs before (u,i,v), and (u,s) before (u,v), since $s < i \leqslant v$. Then (r,s,i) and (u,i,v) occur before (i,j,k), while (r,s) and (u,v) occur before (j,k), since $r < i \leqslant j$, $u < i \leqslant j$; and the result is thus established.

Turning to the equations (5) and (6), we first show that, on a characteristic, we cannot have $dx = 0$. For suppose that $dx = 0$. Then (5) and (6) become

$$dt_{ij} = \sum_{k=1}^{m} t_{ijk}\, dy_k \quad (i \leqslant j;\ i,j = 1,\dots, m), \tag{7}$$

$$ds_i = \sum_{k=1}^{m} s_{ij}\, dy_j \quad (i = 1,\dots, m). \tag{8}$$

Now, if we arrange the equations (7) in the order of the 'sequence of pairs' above and the terms on the right in the order of the 'sequence of triplets', we see that the matrix of the coefficients of the t_{ijk} on the right is a matrix with $\frac{1}{2}m(m+1)$ rows, and $\frac{1}{6}m(m+1)(m+2)$ columns. By (B) above, each row consists of the elements $dy_1, dy_2,\dots, dy_m$, in that order, but interspersed with zeros; while, by (C) above, in each row the element dy_k occurs at least one place further to the right than it does in the preceding row. This matrix is certainly of rank $\frac{1}{2}m(m+1)$, unless $dy_1 = \dots = dy_m = 0$. For, if not, if dy_k be the non-zero element with highest suffix, we can select a determinant of $\frac{1}{2}m(m+1)$ rows, having dy_k in the leading diagonal and zero above the leading diagonal, which is non-zero.

Similarly, we show that the matrix of the s_{ij} on the right of (8) is of rank m unless $dy_1 = \dots = dy_m = 0$. Thus the right-hand sides of (7) and (8) are linearly independent forms in the derivatives of the third order unless $dy_1 = \dots = dy_m = 0$, which is inadmissible if $dx = 0$.

We must therefore have $dx \neq 0$; and to simplify the calculations, we consider a different system, equivalent to (5) and (6). Fixing the index i, we may remove from (5) the restriction $i \leqslant j$, which was imposed only to avoid duplication. Then, multiplying (5) by $-(dy_j - S_j\, dx)$, summing on the index j, multiplying (6) by dx, and adding, we obtain

$$\begin{aligned} ds_i\, dx - \sum_{j=1}^{m} (dy_j - S_j\, dx)\, dt_{ij} + (dF/dy_i)\, dx^2 \\ = -\sum_{j=1}^{m} (dy_j^2 - S_j\, dy_j\, dx + T_{jj}\, dx^2) t_{ijj} - \\ - \sum\sum_{j<k} (2dy_j\, dy_k - S_k\, dy_j\, dx - S_j\, dy_k\, dx + T_{jk}\, dx^2) t_{ijk} \\ (i = 1,\dots, m) \end{aligned} \tag{9}$$

and clearly the system (3), (5), (6) is entirely equivalent to the system (3), (5), (9), when $dx \neq 0$, once again with the restriction $i \leqslant j$ in (5), to avoid duplication.

Since each distinct s_{ij} occurs in one only of the equations (5), when $dx \neq 0$, we may solve the equations (5) for the $\frac{1}{2}m(m+1)$ derivatives

s_{ij}; and thus we need investigate only the conditions for the right-hand sides of (9) to be linearly dependent forms in the t_{ijk}. Let us arrange the equations (9) in the order $i = 1,..., m$ and arrange the terms on the right, according to the suffixes of the t_{ijk}, in the order of the 'sequence of triplets' above. Then the matrix of the coefficients of the t_{ijk}, on the right, is a matrix of m rows, and $\frac{1}{6}m(m+1)(m+2)$ columns. We see that every row consists of the elements a_{jk}, where

$$-a_{jj} = dy_j^2 - S_j\,dy_j\,dx + T_{jj}\,dx^2,$$

$$-a_{jk} = 2dy_j\,dy_k - S_k\,dy_j\,dx - S_j\,dy_k\,dx + T_{jk}\,dx^2 \quad (j < k),$$

interspersed with zeros; by (C) above, we see that, in every row, these elements occur in the same order: that of the 'sequence of pairs'; and, by (B) above, we see that, in each row, a selected element occurs at least one place farther to the right than in the preceding row.

Once again, this matrix is certainly of rank m unless all its elements are zero. For, if not, selecting the non-zero element farthest to the right in the first row, we could pick out a determinant, with this element in the leading diagonal and zero above the leading diagonal, which would be non-zero.

Thus, on a characteristic, we must have each $a_{jj} = a_{jk} = 0$, i.e.

$$dy_j^2 - S_j\,dy_j\,dx + T_{jj}\,dx^2 = 0 \quad (j = 1,..., m), \tag{10}$$

$$2dy_j\,dy_k - S_k\,dy_j\,dx - S_j\,dy_k\,dx + T_{jk}\,dx^2 = 0 \quad (j \neq k;\, j, k = 1,..., m). \tag{11}$$

Let μ_{j1}, μ_{j2} be the roots of (10), regarded as a quadratic in dy_j/dx, so that

$$\left.\begin{aligned} \mu_{j1}+\mu_{j2} &= S_j \\ \mu_{j1}\mu_{j2} &= T_{jj} \end{aligned}\right\} \quad (j = 1,..., m). \tag{12}$$

The equations (10) and (11) then become

$$(dy_j - \mu_{j1}\,dx)(dy_j - \mu_{j2}\,dx) = 0 \quad (j = 1,..., m), \tag{13}$$

$$(dy_j - \mu_{j1}\,dx)(dy_k - \mu_{k2}\,dx) + (dy_j - \mu_{j2}\,dx)(dy_k - \mu_{k1}\,dx) +$$
$$+\{T_{jk} - (\mu_{j1}\mu_{k2} + \mu_{j2}\mu_{k1})\}\,dx^2 = 0 \quad (j < k;\, j, k = 1,..., m). \tag{14}$$

Let us suppose first that the system (13) and (14) admits a consistent solution in dy_j/dx $(j = 1,..., m)$. Then, by suitable labelling of the roots of (13), we can ensure that this solution is

$$dy_j - \mu_{j1}\,dx = 0 \quad (j = 1,..., m). \tag{15}$$

Substituting in (14), we have the necessary condition for consistency

$$T_{jk} = \mu_{j1}\mu_{k2} + \mu_{j2}\mu_{k1} \quad (j < k). \tag{16}$$

If this condition be satisfied, the equations (14) become

$$(dy_j - \mu_{j1}\,dx)(dy_k - \mu_{k2}\,dx) + (dy_j - \mu_{j2}\,dx)(dy_k - \mu_{k1}\,dx) = 0 \quad (j < k). \tag{17}$$

Suppose first that not every $\mu_{j1} = \mu_{j2}$: for example suppose $\mu_{11} \neq \mu_{12}$. Then, putting $dy_1 = \mu_{11}dx$ in (17) with $j = 1$, we see that we must have (15) as before. Again, putting $dy_1 = \mu_{12}dx$ in (17) with $j = 1$, we obtain

$$dy_k - \mu_{k2}dx = 0 \quad (k = 1,..., m); \tag{18}$$

and thus we have precisely two distinct sets of consistent solutions, of (13) and (17), namely (15) and (18). If $\mu_{j1} = \mu_{j2}$ $(j = 1,..., m)$, it is clear that (13) and (17) admit one set of consistent solutions only, i.e. the two sets are confluent.

Each of the sets of solutions (15) and (18) of the equations (13) and (17) which, when the conditions (16) are satisfied, are equivalent to (10) and (11), leads to a system of characteristics of the equation (1). For, substituting for $dy_1,..., dy_m$ from (15) in (9), taking account of (10) and (11), using (12) to express the values of the S_j $(j = 1,..., m)$, and dividing by dx, we obtain (remembering that $t_{ij} = t_{ji}$)

$$ds_i + \sum_{j=1}^{m} \mu_{j2}dt_{ij} + (dF/dy_i)dx = 0 \quad (i = 1,..., m). \tag{19}$$

And these, in view of the definition of (dF/dy_i), are total differential equations in the variables making up the element of contact of the second order, containing $ds_1,..., ds_m$, and independent of the integral on which we suppose the multiplicity to lie, as required by the definition. Substituting also from (15) in (3), we see that, when the conditions (16) are satisfied, we have two systems of characteristics, distinct unless every $\mu_{j1} - \mu_{j2} = 0$ $(j = 1,..., m)$, confluent in the contrary case, one system being defined by the equations

$$\left.\begin{aligned} dy_j - \mu_{j1}dx &= 0 \quad (j = 1,..., m) \\ dz - \Big(p + \sum_{j=1}^{m} \mu_{j1}q_j\Big)dx &= 0 \\ dp + \Big(F - \sum_{j=1}^{m} \mu_{j1}s_j\Big)dx &= 0 \\ dq_i - \Big(s_i + \sum_{j=1}^{m} \mu_{j1}t_{ij}\Big)dx &= 0 \quad (i = 1,..., m) \\ ds_i + \sum_{j=1}^{m} \mu_{j2}dt_{ij} + (dF/dy_i)dx &= 0 \quad (i = 1,..., m) \end{aligned}\right\} \tag{20}$$

and the other by permuting μ_{j1} and μ_{j2} $(j = 1,..., m)$ in (20), μ_{j1} and μ_{j2} being defined by (12). It is easy to see, exactly as in the classical case of two independent variables, that every integral of (1) is a locus of characteristics of both systems.

It remains to find the necessary and sufficient conditions, in terms

of the given equation, for the equations (16) to be satisfied. Consider the symmetric matrix

$$M \equiv \begin{bmatrix} 1 & \tfrac{1}{2}S_1 & \tfrac{1}{2}S_2 & . & . & . & . & \tfrac{1}{2}S_m \\ \tfrac{1}{2}S_1 & T_{11} & \tfrac{1}{2}T_{12} & . & . & . & . & \tfrac{1}{2}T_{1m} \\ \tfrac{1}{2}S_2 & \tfrac{1}{2}T_{12} & T_{22} & . & . & . & . & \tfrac{1}{2}T_{2m} \\ \vdots & \vdots & & & & & & \vdots \\ \tfrac{1}{2}S_m & \tfrac{1}{2}T_{1m} & . & . & . & & \tfrac{1}{2}T_{m-1m} & T_{mm} \end{bmatrix}$$

whose rank defines the *rank* [(5) 112] of the equation (1). Putting in the values (12) for S_j and T_{jj} $(j = 1,..., m)$, and the values (16) for T_{jk} $(j \neq k)$, we see that the matrix M is then equal to the matrix product

$$\begin{bmatrix} 1 & 1 \\ \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \\ \vdots & \vdots \\ \mu_{m1} & \mu_{m2} \end{bmatrix} \times \begin{bmatrix} \tfrac{1}{2} & \tfrac{1}{2}\mu_{12} & . & . & . & \tfrac{1}{2}\mu_{m2} \\ \tfrac{1}{2} & \tfrac{1}{2}\mu_{11} & . & . & . & \tfrac{1}{2}\mu_{m1} \end{bmatrix}.$$

The rank of M cannot, therefore, exceed two; and, by considering the minors

$$\begin{vmatrix} 1 & \tfrac{1}{2}S_i \\ \tfrac{1}{2}S_i & T_{ii} \end{vmatrix} = \begin{vmatrix} 1 & \tfrac{1}{2}(\mu_{i1}+\mu_{i2}) \\ \tfrac{1}{2}(\mu_{i1}+\mu_{i2}) & \mu_{i1}\mu_{i2} \end{vmatrix} = -\tfrac{1}{4}(\mu_{i1}-\mu_{i2})^2 \qquad (i = 1,..., m),$$

we see that, if not every $\mu_{i1}-\mu_{i2}$ is zero, so that there are two distinct systems of characteristics of the second order, the rank of M (i.e. the rank of (1)) is 2. If $\mu_{i1}-\mu_{i2} = 0$ $(i = 1,..., m)$, so that there is one system of characteristics only, the two matrices written above, of which M is the product, have respectively two identical rows only, and two identical columns only. Each is therefore of rank 1; and M, which has a non-zero element, is therefore of rank 1.

Thus we see that in order that the equation (1) may admit two distinct systems of characteristics of the second order, it is necessary that it be of rank 2; and in order that it may admit one system oniy, it is necessary that it be of rank 1. We shall now show that these conditions are also sufficient.

Suppose that the matrix M above is of rank not exceeding 2. We may still define μ_{j1}, μ_{j2} $(j = 1,..., m)$ to be the roots of the quadratic equations (10) in the dy_j/dx $(j = 1,..., m)$, so that S_j and T_{jj} $(j = 1,..., m)$ are still expressed by (12). Then, expressing the fact that, by the hypothesis regarding M, the minor formed from the first, $(j+1)$th and

$(k+1)$th $(j \neq k)$ rows and columns of M is zero, and, using (12), we have

$$\begin{vmatrix} 1 & \frac{1}{2}(\mu_{j1}+\mu_{j2}) & \frac{1}{2}(\mu_{k1}+\mu_{k2}) \\ \frac{1}{2}(\mu_{j1}+\mu_{j2}) & \mu_{j1}\mu_{j2} & \frac{1}{2}T_{jk} \\ \frac{1}{2}(\mu_{k1}+\mu_{k2}) & \frac{1}{2}T_{jk} & \mu_{k1}\mu_{k2} \end{vmatrix} \quad (j,k = 1,\dots,m;\ j \neq k)$$

$$= -\tfrac{1}{4}\{T_{jk}-(\mu_{j1}\mu_{k2}+\mu_{j2}\mu_{k1})\}\{T_{jk}-(\mu_{j1}\mu_{k1}+\mu_{j2}\mu_{k2})\}$$

$$= 0. \tag{21}$$

Suppose, firstly, that at least one of the equations (10) has distinct roots. By suitable labelling, we may suppose that it is the first, so that $\mu_{11}-\mu_{12} \neq 0$. Then, by interchanging where necessary the labels of any other pair of roots μ_{k1} and μ_{k2} $(k > 1)$, we can ensure that each condition (21) in which $j = 1$ becomes

$$T_{1k} = \mu_{11}\mu_{k2}+\mu_{12}\mu_{k1} \quad (k = 2,\dots,m). \tag{22}$$

Let us now equate to zero the determinant formed from the first, second, and $(j+1)$th rows $(j > 1)$ and the first, second, and $(k+1)$th columns $(k > 1;\ k \neq j)$ of M. We then have, using (12) and (22),

$$\begin{vmatrix} 1 & \frac{1}{2}(\mu_{11}+\mu_{12}) & \frac{1}{2}(\mu_{k1}+\mu_{k2}) \\ \frac{1}{2}(\mu_{11}+\mu_{12}) & \mu_{11}\mu_{12} & \frac{1}{2}(\mu_{11}\mu_{k2}+\mu_{12}\mu_{k1}) \\ \frac{1}{2}(\mu_{j1}+\mu_{j2}) & \frac{1}{2}(\mu_{11}\mu_{j2}+\mu_{12}\mu_{j1}) & \frac{1}{2}T_{jk} \end{vmatrix} \quad (j,k = 2,\dots,m;\ j \neq k)$$

$$= -\tfrac{1}{8}(\mu_{11}-\mu_{12})^2\{T_{jk}-(\mu_{j1}\mu_{k2}+\mu_{j2}\mu_{k1})\}$$

$$= 0. \tag{23}$$

Thus, since $\mu_{11}-\mu_{12} \neq 0$, we once again have, from (22) and (23), the conditions (16); and we have seen that, in these circumstances, M is of rank 2 and there are two distinct systems of characteristics of the second order.

Secondly, suppose that

$$\mu_{j1}-\mu_{j2} = 0 \quad (j = 1,\dots,m).$$

Then the condition (21) gives

$$T_{jk} = 2\mu_{j1}\mu_{k1} \quad (j \neq k);$$

and thus, once again, we have the conditions (16) when we write $\mu_{j2} = \mu_{j1}$ $(j = 1,\dots,m)$. In this case, we have shown that M is of rank 1, and that there is one system of characteristics of the second order only.

We therefore see that in order that the equation may admit two distinct systems of characteristics of the second order, it is necessary and sufficient that (1) be of rank 2; and in order that it may admit one system only, it is necessary and sufficient that it be of rank 1.

We have seen that an equation of the second order with $m+1$ inde-

pendent variables, of any form, can be reduced to the form (1) by a simple change of independent variables. But I have shown (loc. cit.) that the rank of an equation of the second order is invariant under any contact transformation. Thus we may at once extend these results to equations of any form. The results for characteristics of higher order than the second are entirely similar, and obtained in exactly the same way. We may summarize the whole matter in the theorem:

THEOREM. *A partial differential equation of the second order, in one dependent and any number of independent variables, admits two distinct systems of one-dimensional characteristics, of the kind defined earlier, of the second and all higher orders, if it be of rank* 2. *It admits one system only, i.e. the two systems are confluent, if it be of rank* 1; *while there are no characteristics of this kind if it be of rank* 3 *or more.*

The extension of Darboux's method to equations of rank 2 or 1 can then be made, on exactly the same lines as I have already developed it for equations with three independent variables (**4**).

There is an interesting link between the theory of one-dimensional characteristics, which we have developed, and the standard theory of the m-dimensional characteristics (known as *Monge-characteristics*) of an equation with $m+1$ independent variables, developed by Beudon (**1**). It is a standard result that, corresponding to a *known* integral of (1), the condition for the m-dimensional hyper-surface ('surface' if $m = 2$)

$$\psi(x, y_1, \ldots, y_m) = 0$$

to be a Monge-characteristic of (1), contained in the integral, is, with our notation,

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \sum_{i=1}^{m} S_i \frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y_i} + \sum_{i\leqslant k}\sum T_{ik}\frac{\partial\psi}{\partial y_i}\frac{\partial\psi}{\partial y_k} = 0, \tag{24}$$

where we suppose that z and its derivatives, in S_i etc., are replaced by their values corresponding to the integral of (1). But the condition that the matrix M above shall be of rank 2 or 1 is precisely the condition for the left-hand side of (24) to be decomposed into factors, i.e. for (24) to become

$$\left(\frac{\partial\psi}{\partial x} + \sum_{i=1}^{m}\mu_{i1}\frac{\partial\psi}{\partial y_i}\right)\left(\frac{\partial\psi}{\partial x} + \sum_{i=1}^{m}\mu_{i2}\frac{\partial\psi}{\partial y_i}\right) = 0, \tag{25}$$

on replacing S_i, T_{ik} by their values given by (12) and (16).

Now the curves associated with the Cauchy-characteristics of (24) are known as the 'bi-characteristics' of (1). If the left-hand side of (24) be irreducible, these bi-characteristics, corresponding to any integral

of (1), form a complex of curves in the space of $m+1$ dimensions, depending, in general, upon $2m-1$ parameters. But, if (24) is reducible and decomposes into (25), the bi-characteristics form two congruences, each depending upon m parameters, one curve of each congruence passing through each point of the $(m+1)$-space. Thus we have the curious result that, if the bi-characteristics associated with any integral of an equation of the second order form a complex of curves, the equation does not admit one-dimensional characteristics; but, if the bi-characteristics form two congruences (which may be distinct or confluent), then two families of one-dimensional characteristics exist. Furthermore, we may assert that, in this case, the curves associated with these latter coincide with the bi-characteristics. For the Cauchy-characteristics of the two linear equations of the first order, represented by (25), are defined by the equations

$$\frac{dx}{1} = \frac{dy}{\mu_{i1}} \quad (i = 1,..., m),$$

$$\frac{dx}{1} = \frac{dy}{\mu_{i2}} \quad (i = 1,..., m),$$

in which μ_{i1} and μ_{i2} $(i = 1,..., m)$ are now functions of x, $y_1,..., y_m$ since z and its derivatives are supposed replaced by their values corresponding to the integral of (1). Thus we see from (15) and (18) that the two families of Cauchy-characteristics of the two first-order equations represented by (25) are precisely the two families of curves associated with those characteristics, contained in the integral of (1), which are defined by (20) above, and by a similar system of equations with μ_{i1}, μ_{i2} permuted.

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CONNEXIONS FOR PARALLEL DISTRIBUTIONS IN THE LARGE (II)

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1. Introduction

In a previous paper† it was shown that on a differentiable manifold an affine connexion always exists globally with respect to which one or more given distributions are parallel. It was also shown that, if the given system of distributions is integrable, the connexion can be chosen to be symmetric, i.e. torsion-free.

The present paper continues the study of global connexions related to given distributions, and properties such as relative parallelism and path-parallelism with respect to a connexion are defined and considered.‡ A number of existence theorems for global connexions are given, these connexions being torsion-free whenever possible, and in each case a formula for the simplest connexion having the desired properties is constructed. All such formulae are expressed in terms of the projection tensors associated with the given distributions, and are simplified by means of a convenient notation for the various projections of a tensor. With this notation the calculations are very similar to those resulting from the use of forms and special frames. The present method has the advantage, however, that the resulting formulae for tensors and connexions are expressed in relation to a general coordinate system: no transformation from a special to a general frame is necessary, and formulae are in a convenient form for subsequent applications to special problems.

In a later paper it will be shown how some of the present results lead to a definition of 'torsional derivation' and the construction of new concomitants for an almost complex structure. Certain holonomic properties of some of the connexions given here will also be discussed in

† A. G. Walker, *Quart. J. of Math.* (Oxford) (2) 6 (1955) 301–8. This paper will be referred to as (**1**).

‡ Some of these properties are closely related to properties of non-holonomic submanifolds studied in the 1930's by Vranceanu, P. Dienes, and others. This earlier work was all local, however, and was not concerned with the present problem of establishing existence theorems and constructing global connexions having the desired properties.

another paper, where it will be shown how they are related to the foliation groups of Ehresmann.

It will generally be assumed that the given structure (manifold and distributions) is of class C^∞, and it will then be seen that the connexions we construct are all of the same class C^∞. If, however, the given structure is analytic and if the manifold is such that there exists globally an analytic connexion, then the constructed connexions will be analytic.

2. Projection tensors

Let M be an n-dimensional manifold and D', D'' two complementary distributions over M, of dimensions r', r'', where $r'+r'' = n$. At any point $x \in M$ the r'-plane and r''-plane belonging to D' and D'' are denoted by D'_x and D''_x; these are sub-spaces of the tangent plane T_x and are complementary in the sense that they are disjoint and their sum is T_x.

A vector $u \in T_x$ decomposes to give $u = u'+u''$, where $u' \in D_x$ and $u'' \in D''_x$. The *projection tensors* at x associated with D', D'' are the endomorphisms $\dot{a}$, $\ddot{a}$ of T_x given by

$$\dot{a}u = u', \qquad \ddot{a}u = u''.$$

They are mixed second-order tensors of ranks r', r'' and satisfy the usual identities

$$\dot{a}^2 = \dot{a}, \quad \ddot{a}^2 = \ddot{a}, \quad \dot{a}\ddot{a} = \ddot{a}\dot{a} = 0, \quad \dot{a}+\ddot{a} = I. \tag{1}$$

These tensors are defined at every point of M and form tensor fields of the same class as the given structure (M, D', D'').

The vectors $u' = \dot{a}u$ and $u'' = \ddot{a}u$ are the projections of the contravariant vector u. We can also define the projections of a covariant vector and of any tensor, but this is done most simply in terms of their components relative to a general local coordinate system. If $\dot{a}^i_j$ and $\ddot{a}^i_j$ are the components of $\dot{a}$ and $\ddot{a}$, then for a contravariant vector (u^i) the two projections are $\dot{a}^i_p u^p$ and $\ddot{a}^i_p u^p$, and for a covariant vector (v_i) there are two projections $\dot{a}^p_i v_p$ and $\ddot{a}^p_i v_p$. For a more general tensor there are two projections for each suffix, so that the tensor may be partly or wholly projected in a number of ways. Every projection is a tensor of the same kind, and a convenient notation is to attach one or two primes to a suffix to denote projection by $\dot{a}$ or $\ddot{a}$. We should thus write

$$u^{i'} = \dot{a}^i_p u^p, \quad u^{i''} = \ddot{a}^i_p u^p, \quad v_{i'} = \dot{a}^p_i v_p, \quad v_{i''} = \ddot{a}^p_i v_p,$$

and for a tensor $T^{ij\cdot}_{k\cdot\cdot}$ two of the projections are

$$T^{i'j\cdot}_{k\cdot\cdot} = \dot{a}^i_p T^{pj\cdot}_{k\cdot\cdot}, \qquad T^{i''j'\cdot}_{k''\cdot\cdot} = \ddot{a}^i_p \dot{a}^j_q \ddot{a}^r_k T^{pq\cdot}_{r\cdot\cdot}.$$

Because of the identity $\dot{a}+\ddot{a} = I$, there is a relation between the projections for each suffix, which can be written symbolically $i'+i'' = i$. Thus

$$u^{i'}+u^{i''} = u^i, \quad v_{i'}+v_{i''} = v_i, \quad T^{i'j}_k+T^{i''j}_k = T^{ij}_k, \quad \text{etc.}$$

The summation convention will operate regardless of primes: thus

$$v_i u^{i'} = v_i a^i_p u^p = v_{i'} u^i = v_{i'} u^{i'}$$

since $\dot{a}^2 = \dot{a}$. Also, because $\ddot{a}\dot{a} = \ddot{a}\dot{a} = 0$, we have

$$v_{i'} u^{i''} = 0, \qquad v_{i''} u^{i'} = 0.$$

Suppose now that L is any connexion. Denoting covariant differentiation with respect to L by a solidus, we have $\delta^i_{j/k} = 0$, and, from $\dot{a}+\ddot{a} = I$,

$$\dot{a}^i_{j/k}+\ddot{a}^i_{j/k} = 0. \tag{2}$$

Also, from $\dot{a}\ddot{a} = 0$ and (2),

$$\dot{a}^i_{p/k}\ddot{a}^p_j = -\dot{a}^i_p \ddot{a}^p_{j/k} = \dot{a}^i_p \dot{a}^p_{j/k}. \tag{3}$$

With our projection notation it is convenient to adopt the convention of projecting *after* differentiating, so that, for example, $\dot{a}^i_{p/k}\ddot{a}^p_j$ will be written $\dot{a}^i_{j''/k}$. Then (3) gives

$$\dot{a}^i_{j''/k} = \dot{a}^{i'}_{j/k}, \tag{4}$$

and from $\ddot{a}\dot{a} = 0$ it can similarly be deduced that

$$\dot{a}^i_{j'/k} = \dot{a}^{i''}_{j/k}. \tag{5}$$

I shall now write

$$a^i_{jk}(L) = \dot{a}^{i'}_{j/k}+\ddot{a}^{i''}_{j/k} \tag{6}$$

and observe that $a^i_{jk}(L)$ is unaltered when we interchange D' and D'', i.e. $\dot{a}$ and $\ddot{a}$. From (2), (4), and (5) we find

$$\left.\begin{aligned} a^{i'}_{jk}(L) &= a^i_{j''k}(L) = \dot{a}^{i'}_{j/k} \\ a^{i''}_{jk}(L) &= a^i_{j'k}(L) = -\dot{a}^{i''}_{j/k} \\ a^{i'}_{j'k}(L) &= a^{i''}_{j''k}(L) = 0 \end{aligned}\right\}. \tag{7}$$

If Γ is another connexion and $L = \Gamma+X$, so that $X = (X^i_{jk})$ is a tensor, a simple calculation gives the relation

$$a^i_{jk}(L) = a^i_{jk}(\Gamma)-X^{i'}_{j''k}-X^{i''}_{j'k}. \tag{8}$$

3. Integrability and parallelism

Some of the expressions occurring in (1) can now be written more concisely. For example, the condition for D' to be integrable is†

$$a^i_{[j'k']}(\Gamma) = 0, \tag{9}$$

† As usual, $T_{[ij]}$ is written for $\frac{1}{2}(T_{ij}-T_{ji})$ and $T_{(ij)}$ for $\frac{1}{2}(T_{ij}+T_{ji})$.

where Γ is any symmetric connexion, this being obtained directly from the fact that the system of partial differential equations $\dot{a}^p_i \partial_p f = 0$ are required to be completely integrable.

The condition for D' to be parallel with respect to a connexion L can now be written
$$a^i_{j'k}(L) = 0. \tag{10}$$
This can be obtained immediately as follows. If δ denotes the absolute differential corresponding to dx^i and if (u^i) is a vector which lies in D' while undergoing parallel displacement, then $\delta u^i = 0$ and $\delta(\ddot{a}^i_p u^p) = 0$. Hence $(\delta \ddot{a}^i_p)u^p = 0$ for all dx^i and all vectors (u^i) in D', i.e. $\ddot{a}^i_{j'/k} = 0$, and (10) follows from (2) and (7).

Similarly the condition for D'' to be parallel with respect to L is $a^i_{j''k}(L) = 0$, and combining this with (10) we see that the condition for both D' and D'' to be parallel with respect to L is
$$a^i_{jk}(L) = 0. \tag{11}$$
In (**1**) it was proved that there is a global connexion L which satisfies (10) and is symmetric when D' is integrable. It was also proved that there is a global connexion which satisfies (11) and is symmetric when both D' and D'' are integrable. To obtain expressions for these connexions we first choose a global symmetric connexion Γ of class C^∞ (or C^ω if the given structure is analytic and if it is known that an analytic connexion exists); it is well known that this can always be done on a manifold of class C^∞. Then, if we write $L = \Gamma + T$ and use (8), equations (10) for L become equations for T,
$$T^{i''}_{j'k} = a^i_{j'k}(\Gamma).$$
For simplicity we choose $T^{i'}_{jk} = 0$, $T^{i''}_{j''k''} = 0$, and, for symmetry,
$$T^{i''}_{j'k''} = T^{i''}_{k''j'} = a^i_{k''j'}(\Gamma).$$
Then
$$T^i_{jk} = a^i_{j'k}(\Gamma) + a^i_{k''j'}(\Gamma). \tag{12}$$
We now see from (9) that $T^i_{jk} = T^i_{kj}$ when D' is integrable. Thus $L = \Gamma + T$, where T is given by (12), is a connexion which satisfies (10) and is symmetric when D' is integrable. This is the connexion given in (**1**) though expressed differently.

By a similar method it can be shown that the connexion $L = \Gamma + S$, where
$$S^i_{jk} = 2a^i_{(jk)}(\Gamma) - a^i_{k'j'}(\Gamma) - a^i_{k''j''}(\Gamma), \tag{13}$$
satisfies (11) and is symmetric when both D' and D'' are integrable. This again is equivalent to a connexion given in (**1**).

We observe that the connexions $\Gamma + T$ and $\Gamma + S$ have class C^∞, and

are analytic if the given structures and Γ are analytic. These connexions are defined globally because the connexion Γ and the tensors on the right in (12) and (13) are global. Similar results about class and global character will hold for other connexions constructed in this paper but will not be mentioned explicitly.

The torsion tensor associated with the connexion L is $H^i_{jk} = L^i_{[jk]}$, so that for $L = \Gamma + S$ we have the torsion tensor $H^i_{jk} = S^i_{[jk]}$ since Γ is symmetric. Hence from (13) we find

$$H^i_{jk} = a^i_{[j'k']}(\Gamma) + a^i_{[j''k'']}(\Gamma). \tag{14}$$

It is easily verified that this tensor is in fact independent of the symmetric connexion Γ and is equivalent, to within a numerical factor, to the torsion defined by Nijenhuis† for an almost complex structure. Such a structure is determined by a tensor h satisfying $h^2 = -I$, and the relation between our $\dot{a}$ and this h is $\dot{a} = \frac{1}{2}(I + ih)$. The torsion as defined here appears naturally as the torsion of certain connexions associated with D' and D'' and is not merely defined as a tensor which vanishes when the given structure is integrable. It has, of course, the latter property, for the two terms on the right in (14) vanish when D' and D'' respectively are integrable. Conversely it follows from (7) that

$$a^i_{[j'k']}(\Gamma) = H^{i''}_{jk}, \qquad a^i_{[j''k'']}(\Gamma) = H^{i'}_{jk},$$

so that D' and D'' are integrable when $H^i_{jk} = 0$.

4. Relative parallelism

An *integral curve* of a distribution D' is a differentiable curve in M with the property that the tangent vector at any point x of the curve lies in D'_x. If D' is integrable, the planes of D' are tangent to a system of submanifolds, or *laminations* (foliations), and M is said to have a *laminated* (foliated) structure. In this case the integral curves of D' are the differentiable curves lying in the laminations.

I shall say that a vector field u is *parallel relative to* D' (with respect to a connexion L) if, for any points x, y on an integral curve of D', the vectors u_x, u_y are parallel relative to this curve.‡ If D' is integrable and x, y are any two points on one of the laminations determined by D', then u_x, u_y are parallel relative to any differentiable arc xy lying in the lamination. The fact that these vectors need not be parallel relative to an arc xy that does not lie in the lamination shows that

† *Proc. Kon. Nederlandse Ak.* (A), 58 (1955) 390–403, § 3.

‡ When referring to parallel vectors along a curve we find it convenient to describe the vectors as parallel *relative to* the curve, and *with respect to* the connexion.

relative parallelism is weaker than parallelism. It is clear that a parallel vector field is parallel relative to every distribution.

In terms of local coordinates and the projection tensors associated with D' and a complementary distribution D'', the vector field (u^i) is parallel relative to D' if $\delta u^i = 0$ for all dx^i lying in D'. Since $\delta u^i = u^i_{|j}\, dx^j$, this condition becomes

$$u^i_{|j'} = 0. \tag{15}$$

Relative parallelism can be extended from a vector field to any distribution, and I shall say that a distribution D^* is parallel relative to D' (with respect to L) if the planes of D^* at points of any integral curve of D' are parallel relative to this curve. If (b^i_j) is a projection tensor such that the vectors of D^* are given by $b^i_j u^j = u^i$, we require $\delta(b^i_j u^j) = 0$ when $\delta u^i = 0$ for all u^i in D^* and all dx^i in D'. Hence $b^i_{j|k}\, u^i\, dx^k = 0$ for all u^i in D^* and dx^i in D', i.e.

$$b^i_{p|q}\, b^p_j\, \dot{a}^q_k = 0. \tag{16}$$

This, then, is the condition to be satisfied by (b^i_j) for the associated distribution D^* to be parallel relative to D' with respect to L.

5. Special relations

In this section we consider various ways in which a connexion can be related to a given distribution D' or complementary pair D', D''. If only D' is given, then D'' is taken to be any complementary distribution.

(I) I shall say that D' is *semi-parallel* with respect to L if it is parallel relative to itself. The condition for this is given at once by (16) with $\dot{a}$ in place of b, and is therefore equivalent to

$$a^i_{j'k'}(L) = 0. \tag{17}$$

Clearly, D' is semi-parallel if it is parallel. Also, D' is integrable if it is semi-parallel with respect to a symmetric connexion.

(II) We may have D'' parallel relative to D'. The condition for this is given by (16) with $\ddot{a}$ in place of b and is therefore equivalent to

$$a^i_{j''k'}(L) = 0. \tag{18}$$

This does *not* imply any integrability conditions when L is symmetric.

(III) Interchanging D' and D'' in (II), we see that D' is parallel relative to D'' if $a^i_{j'k''}(L) = 0$. This can be combined with (18), and D', D'' are each parallel relative to the other if

$$a^i_{j''k'}(L) = 0, \qquad a^i_{j'k''}(L) = 0. \tag{19}$$

As will be seen from Theorem 2 of § 6, these again do not imply any integrability condition when L is symmetric.

(IV) We may have D' parallel and at the same time D'' parallel relative to D'. The conditions for this are (10) and (18), and combining these we see that they can be written

$$a^i_{j'k}(L) = 0, \qquad a^i_{jk'}(L) = 0. \tag{20}$$

These, of course, require D' to be integrable if L is symmetric; D'' need not, however, be integrable.

(V) Another property a distribution D' may have with respect to a connexion is that of being *path-parallel*. A given connexion L determines a system of auto-parallels, or *paths* (geodesics in the case of a Riemannian manifold), such that for any point x and tangent vector u at x, there is just one path through x tangent to u. We say that D' is *path-parallel* with respect to L if, for every point x in M and vector u in D'_x, the path determined by x and u is an integral curve of D'.

Writing $u^i = dx^i/dt$ for the tangent vector to a path, we have $\delta u^i = 0$, and we require

$$\delta(\dot{a}^i_j u^j) = \dot{a}^i_{j/k} u^j u^k \, dt = 0$$

for all vectors u^i in D'. Hence $\dot{a}^i_{j'k'} u^j u^k = 0$ for all u^i, i.e. from (7),

$$a^i_{(j'k')}(L) = 0. \tag{21}$$

This, then, is the condition for D' to be path-parallel with respect to L.

From (17) we see that, if D' is semi-parallel (with respect to L), then it is path-parallel. Path-parallelism is weaker than semi-parallelism, because, for example, D' can be path-parallel with respect to a symmetric connexion without being integrable. If, however, D' is both integrable and path-parallel with respect to L, then D' is also semi-parallel with respect to the symmetric part of L. To see this we write $L = \Gamma + X$, where Γ is symmetric and X is skew-symmetric. Then, from (8),

$$a^i_{j'k'}(L) = a^i_{j'k'}(\Gamma) - X^{i''}_{j'k'},$$

and (21) becomes $a^i_{(j'k')}(\Gamma) = 0$. Since D' is given integrable,

$$a^i_{[j'k']}(\Gamma) = 0 \quad \text{and hence} \quad a^i_{j'k'}(\Gamma) = 0,$$

i.e. D' is semi-parallel with respect to Γ.

We can further prove the lemma:

Lemma. *If L is symmetric and if D' is integrable, path-parallel, and parallel relative to D'' with respect to L, then D' is parallel.*

Since L is symmetric and D' is integrable, we have $a^i_{[j'k']}(L) = 0$, and hence from (21), since D' is path-parallel, $a^i_{j'k'}(L) = 0$. For D' to be

parallel relative to D'' we have (18) with D' and D'' interchanged, i.e. $a^{i}_{j'k''}(L) = 0$, and combining this with $a^{i}_{j'k'}(L) = 0$ we have $a^{i}_{j'k}(L) = 0$. This is the condition for D' to be parallel, which proves the lemma.

6. Special connexions

We are now in a position to construct various connexions related to a given distribution D' or complementary pair D', D''. Choosing a global symmetric connexion Γ as before, we find the conditions to be satisfied by the tensor $L-\Gamma$ in order that L should have the desired property. The conditions turn out to be algebraic, and in each case we find the solution which is simplest in relation to the chosen Γ.

For the remainder of this paper I shall write a^{i}_{jk} for $a^{i}_{jk}(\Gamma)$.

THEOREM 1. *For any complementary distributions D', D'' there is a global symmetric connexion with respect to which both distributions are path-parallel.*

From (21) and (8), D' and D'' are both path-parallel with respect to $L = \Gamma + A$ if A satisfies the algebraic equations

$$A^{i''}_{(j'k')} = a^{i}_{(j'k')}, \qquad A^{i'}_{(j''k'')} = a^{i}_{(j''k'')}. \tag{22}$$

In addition to these restrictions we want A to be symmetric, and the simplest solution is therefore

$$A^{i}_{jk} = a^{i}_{(j'k')} + a^{i}_{(j''k'')}. \tag{23}$$

The connexion $\Gamma + A$, where A is given by (23), satisfies the requirements of the theorem, which is therefore proved.

This theorem can be generalized to any system of disjoint distributions, where every sum of distributions of the system is required to be path-parallel. Here again it is not difficult to prove that a global symmetric connexion exists as required.

It will now be shown that further conditions can be imposed on the connexion in Theorem 1.

THEOREM 2. *For any complementary distribution D', D'' there is a global symmetric connexion with respect to which each distribution is path-parallel and parallel relative to the other.*

We now want L to be symmetric and satisfy (19) in addition to the previous conditions. Writing $L = \Gamma + B$, we require B to be symmetric and to satisfy (22) (with B in place of A) and also, from (8) and (19),

$$B^{i'}_{j''k'} = a^{i}_{j''k'}, \qquad B^{i''}_{j'k''} = a^{i}_{j'k''}.$$

The simplest solution is found by taking $B^{i'}_{j'k'} = B^{i''}_{j''k''} = 0$, and can be written

$$B^{i}_{jk} = 2a^{i}_{(jk)} - a^{i}_{(j'k')} - a^{i}_{(j''k'')}. \tag{24}$$

It can be verified that the connexion $\Gamma+B$ with B given by (24) satisfies the requirements of the theorem.

We observe that B in (24) is the symmetric part of S in (13), i.e. $B^i_{jk} = S^i_{(jk)}$, and therefore $B = S$ when D' and D'' are both integrable since S is then symmetric.

If we relax the requirements that D' and D'' should be path-parallel and seek a symmetric connexion with respect to which D' and D'' are parallel relative to each other, the above solution $\Gamma+B$ is not the simplest, which is easily seen to be $\Gamma+C$, where

$$C^i_{jk} = 2a^i_{(j'k'')}+2a^i_{(j''k')}. \tag{25}$$

Returning to the connexion $L = \Gamma+B$ constructed to prove Theorem 2, it is notable as being probably the simplest global connexion (related to a chosen Γ) which is symmetric and is associated with any two complementary distributions, without assuming any particular properties such as integrability. With this connexion L we have from (19) and (21),

$$a^i_{j'k''}(L) = a^i_{j''k'}(L) = a^i_{(j'k')}(L) = a^i_{(j''k'')}(L) = 0, \tag{26}$$

and it follows that $a^i_{jk}(L)$ is skew-symmetric. This tensor is in fact the torsion tensor defined in § 3, for, since the present L is symmetric, we have (14) with L in place of Γ and from (26) we find

$$H^i_{jk} = a^i_{jk}(L). \tag{27}$$

This simple relation is true for any connexion L which satisfies the conditions of Theorem 2.

If, in Theorem 2, D' is integrable, then by the lemma of § 5, D' is parallel. Thus for any integrable D' and complementary D'' there is a symmetric connexion with respect to which D' is parallel and D'' is path-parallel and parallel relative to D'. This is contained in the following theorem.

THEOREM 3. *For any complementary distributions D', D'' there is a global connexion with respect to which D' is parallel and D'' is path-parallel and parallel relative to D'; this connexion can be chosen formally so that it becomes symmetric when D' is integrable.*

To prove this we construct a connexion L which satisfies (10) and (18), together with (21) with D' and D'' interchanged, i.e. $a^i_{(j''k'')}(L) = 0$. Choosing a symmetric Γ as before and writing $L = \Gamma+V$, we get the equations for V as

$$V^{i''}_{j'k} = a^i_{j'k}, \quad V^{i'}_{j''k'} = a^i_{j''k'}, \quad V^{i'}_{(j''k'')} = a^i_{(j''k'')}.$$

We also want V to be symmetric when D' is integrable, i.e. when

$a^i_{[j'k']} = 0$. For the simplest solution we therefore take $V^{i''}_{j'k'} = V^{i'}_{j''k''} = 0$ and find

$$V^i_{jk} = 2a^i_{(jk)} - a^i_{(j''k'')} - a^i_{k'j'}. \tag{28}$$

With this V the connexion $\Gamma + V$ is found to satisfy all the requirements of Theorem 3.

As expected, V becomes B in (24) when D' is integrable. We also observe that, if D' and D'' are both integrable, then, by the lemma of § 5, D'' is parallel with respect to the symmetric connexion $\Gamma + V$. In this case V and S in (13) become the same tensor.

It should be noted that, if in Theorem 3 we exclude the last requirement, then $\Gamma + V$ is not the simplest solution. The simpler connexion

$$L^i_{jk} = \Gamma^i_{jk} + a^i_{jk} \tag{29}$$

makes both D' and D'' parallel but does not necessarily become symmetric when D' and D'' are integrable. Here Γ is any symmetric connexion, but in the next section the same connexion L appears with Γ as the Christoffel connexion given by a certain metric.

7. Distributions and metrics

Distributions D', D'' are orthogonal with respect to a (Riemannian) metric g if at every point x, $g_{ij} u^i v^j = 0$ for all vectors u in D' and v in D'', the components of g, u, v being relative to some coordinate system in the neighbourhood of x. In projection notation this condition for orthogonality is equivalent to

$$g_{i'j''} = 0 \tag{30}$$

when D', D'' are complementary distributions. We now observe that

For any complementary distribution D', D'' there is a positive-definite metric g with respect to which D', D'' are orthogonal. The class of g is C^∞, and is analytic if the given structure is analytic and if M is known to admit a positive-definite analytic metric.

This follows from the fact that M admits a positive-definite metric, h say, of class C^∞. The required metric is now given by

$$g_{ij} = h_{i'j'} + h_{i''j''}$$

in every coordinate neighbourhood; this symmetric tensor is defined globally, and g is at once seen to be positive-definite. If M, D', D'', and h are analytic, then g is analytic.

Theorem 4. *For any complementary distributions D', D'' which are orthogonal with respect to a metric g, there is a global connexion L with respect to which D' and D'' are parallel and g is constant, i.e. $g_{ij/k} = 0$.*

e write $L = \Gamma + W$, where Γ is now chosen to be the Christoffel

connexion given by g, so that $g_{ij,k} = 0$, a comma denoting covariant differentiation with respect to Γ. We write

$$\dot a_{ij} = g_{ip}\dot a^p_j, \qquad \ddot a_{ij} = g_{ip}\ddot a^p_j, \qquad a^i_{jk} = a^i_{jk}(\Gamma),$$

and establish the identities

$$\dot a_{ij} = \dot a_{ji}, \qquad \ddot a_{ij} = \ddot a_{ji}, \tag{31}$$

$$g_{ip}a^p_{jk} + g_{jp}a^p_{ik} = 0. \tag{32}$$

To prove (31), we have $g_{i''j'} = 0$ from (30) and hence

$$\dot a_{ij} = g_{ij'} = g_{i'j'} = g_{j'i'} = \dot a_{ji}.$$

Similarly, $\ddot a_{ij} = \ddot a_{ji}$. To prove (32) we use the identities (2), (4), (5), and (7) with Γ in place of L. We have

$$\begin{aligned} g_{ip}a^p_{jk} &= g_{ip}(\dot a^{p'}_{j,k} + \ddot a^{p''}_{j,k}) \\ &= g_{i'p}\dot a^p_{j,k} + g_{i''p}\ddot a^p_{j,k}, \quad \text{from (30)}, \\ &= \dot a_{i'j,k} + \ddot a_{i''j,k}, \quad \text{since } g_{ij,k} = 0, \\ &= \dot a_{ji',k} + \ddot a_{ji'',k}, \quad \text{from (31)}, \\ &= g_{jp}(\dot a^p_{i,k} + \ddot a^p_{i,k}) = -g_{jp}a^p_{ik}. \end{aligned}$$

The equations for W are given by $a^i_{jk}(L) = 0$ from (11) and $g_{ij/k} = 0$, and, from (8) and $g_{ij,k} = 0$, these become

$$\left.\begin{aligned} W^{i'}_{j''k} + W^{i''}_{j'k} &= a^i_{jk} \\ g_{ip}W^p_{jk} + g_{jp}W^p_{ik} &= 0 \end{aligned}\right\}. \tag{33}$$

From (7) and (32) we see at once that

$$W^i_{jk} = a^i_{jk}$$

is a solution of these equations, and the theorem is proved.

The connexion $\Gamma + W$ is determined uniquely by D', D'', and g and is the simplest having the desired properties. It is not however the most general connexion having these properties since it can easily be verified that a^i_{jk} is not the only solution of equations (33). This means that further geometrical requirements could be imposed on the connexion in Theorem 4. One condition we cannot, however, impose in general is that of symmetry, for, if L is symmetric, then $g_{ij/k} = 0$ implies that $L = \Gamma$ and hence, from (11), $a^i_{jk} = 0$. If and only if this condition is satisfied is there a symmetric connexion satisfying the requirements of Theorem 4.

A MULTIVARIATE GENERALIZATION OF TCHEBICHEV'S INEQUALITY

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1. A CONSIDERABLE literature has been built upon the simple and basic Tchebichev inequality, most of it concerned with the strengthening of the result when additional information on the variate distribution makes this possible. [For reviews of the literature see Shohat and Tamarkin (**5**), Fréchet (**2**) 130.] A few writers have considered the extension of the result to the multivariate case (potentially useful for theoretical work, although not for practical statistics).

In its conventional form Tchebichev's proposition is that, if a statistical variate x has zero mean and variance v and

$$P = \text{prob}(|x| > \alpha), \tag{1}$$

then

$$P \leqslant v\alpha^{-2}. \tag{2}$$

If the statement is modified to

$$P \leqslant \{v\alpha^{-2}\}, \tag{3}$$

where

$$\{y\} = \begin{cases} y & (0 \leqslant y \leqslant 1), \\ 1 & (y > 1), \end{cases} \tag{4}$$

then it becomes the strongest assertion possible in the absence of any further information on the distribution of x. This follows from the fact that distribution functions can always be found for which the equality sign in (3) is fulfilled. (For $v \leqslant \alpha^2$ consider the distribution whose mass is concentrated at the points $-\alpha-\epsilon$, 0, $\alpha+\epsilon$, where ϵ is an arbitrarily small positive number, in amounts $\frac{1}{2}v\alpha^{-2}$, $1-v\alpha^{-2}$, $\frac{1}{2}v\alpha^{-2}$; for $v > \alpha^2$ consider the distribution with masses $\frac{1}{2}$ at $x = \pm\sqrt{v}$.)

In this paper the inequality is generalized to the case of n variates $x_1, x_2,..., x_n$ with zero means and known second moments v_{jk}, where P is now defined by

$$1-P = \text{prob}(|x_j| \leqslant \alpha_j; j = 1, 2,..., n). \tag{5}$$

The obvious extension of (3),

$$P \leqslant \Big\{\sum_1^n v_{jj}/\alpha_j^2\Big\}, \tag{6}$$

is actually the best possible if the variates are uncorrelated, but may be very inefficient if there is considerable correlation between variates.

The case $n = 2$ has been fully treated by Berge (**1**) and Lal (**3**), whose methods are in part due to Pearson (**4**). Lal shows that

$$P \leqslant \left\{\frac{(v_{11}\alpha_2^2+v_{22}\alpha_1^2)+\sqrt{[(v_{11}\alpha_2^2+v_{22}\alpha_1^2)^2-4v_{12}^2\alpha_1^2\alpha_2^2]}}{2\alpha_1^2\alpha_2^2}\right\}. \tag{7}$$

Berge had previously obtained this result for the special case

$$\alpha_1^2/\alpha_2^2 = v_{11}/v_{22}.$$

Lal gives a result for general n, but his inequality is the sharpest possible only for $n = 2$, when it reduces to (7).

It is true in the multivariate as in the univariate case that substantially sharper results can be obtained if more is known of the distribution than just the first and second moments. For instance, if the x_j were known to be independent, then (6) could be strengthened to

$$P \leqslant 1-\prod_1^n [1-\{v_{jj}/\alpha_j^2\}], \tag{8}$$

which is a great improvement in that the dependence upon n is exponential rather than linear. However, generalization on the basis of the first and second moments alone is a necessary first step in a multivariate theory.

Notation. I shall abbreviate the terms 'positive definite' and 'non-negative definite' to 'p.d.' and 'n.n.d.' respectively, and it is to be understood that matrices designated by either of these terms are symmetric.

2. The lemmas of this section are not essential for an understanding of the main argument and results, and the reader who wishes can proceed directly to § 3.

Lemma 1. *If r, s are positive numbers such that $r+s = 1$, and B_1, B_2 are p.d. matrices, then*

$$M = rB_1^{-1}+sB_2^{-1}-(rB_1+sB_2)^{-1} \tag{9}$$

is n.n.d., and equals zero only if $B_1 = B_2$. In this sense B^{-1} is a convex matrix function of the elements of B if B is p.d.

The matrices

$$R = B_1^{-1}, \quad S = B_2^{-1}, \quad T = (rS+sR)^{-1} \tag{10}$$

are all p.d. We have

$$\begin{aligned} M &= rR+sS-(rR^{-1}+sS^{-1})^{-1} \\ &= \tfrac{1}{2}[rRTT^{-1}+rT^{-1}TR+sSTT^{-1}+sT^{-1}TS-STR-RTS] \\ &= \tfrac{1}{2}[2rs(RTR+STS)+(r^2+s^2-1)(RTS+STR)] \\ &= rs[RTR+STS-RTS-STR] \\ &= rs(R-S)T(R-S), \end{aligned} \tag{11}$$

which is plainly n.n.d. If M were zero, then $\xi'(R-S)T(R-S)\xi$ would be zero for any vector ξ; since T is p.d., this implies $(R-S)\xi = 0$ for all ξ, whence $R = S$ and $B_1 = B_2$.

Lemma 2. *If V and B are arbitrary p.d. matrices, then*

$$\Pi(B) = \mathrm{tr}(VB^{-1}) \tag{12}$$

is a convex function of the elements of B.

We have, for arbitrary, unequal p.d. B_1 and B_2

$$r\Pi(B_1)+s\Pi(B_2)-\Pi(rB_1+sB_2) = \mathrm{tr}(VM), \tag{13}$$

where M is the n.n.d. matrix defined by (9). If M has spectral representation

$$M = \sum \mu_j \eta_j \eta_j', \tag{14}$$

where the μ_j are non-negative and not all zero, then

$$\mathrm{tr}(VM) = \sum \mu_j(\eta_j' V\eta_j) > 0, \tag{15}$$

which proves the assertion.

Lemma 3. *If V is p.d. then*

(i) *there is a unique B which minimizes $\Pi = \mathrm{tr}(VB^{-1})$ subject to the conditions that B be p.d. and have prescribed diagonal elements;*

(ii) *this B satisfies*

$$V = B\Lambda B, \tag{16}$$

where Λ is a diagonal matrix of positive elements λ_j;

(iii) *the minimized value of Π is*

$$\min \Pi = \sum \lambda_j b_{jj}; \tag{17}$$

(iv) *it follows that V has a unique representation of form* (16), *B, Λ having the properties described in* (i), (ii).

Consider the space of the $\frac{1}{2}n(n-1)$ variables b_{jk} $(j > k)$, and let b denote the representative point in this space. Let Γ denote the domain in this space for which B is n.n.d. for the given values of the diagonal elements b_{jj}. It is in the interior of this finite domain, where B is p.d., that b must be chosen.

If the spectral representation of B is

$$B = \sum \beta_j \xi_j \xi_j', \tag{18}$$

then

$$\Pi(B) = \sum \beta_j^{-1}(\xi_j' V \xi_j). \tag{19}$$

As b approaches the boundary of Γ, one or more of the β_j will approach zero, and Π will become indefinitely large since none of the coefficients of the β_j^{-1} in (19) is less than the least eigenvalue of V, which is a fixed positive number.

Since $\Pi(B)$ is finite and continuous inside Γ but approaches plus infinity on the boundary, and Γ is finite, $\Pi(B)$ must reach its minimum in the interior of Γ. The minimum point will be a stationary point since the derivatives of $\Pi(B)$ are also continuous inside Γ. But there is at most one stationary point since, by Lemma 2, $\Pi(B)$ is convex. It follows that there is exactly one stationary point, and that at this point $\Pi(B)$ assumes its minimum value.

We now establish (ii) by equating to zero the differential coefficients $\partial\Pi/\partial b_{jk}$ (see § 3 for details). The λ_j are the Lagrangian multipliers associated with the prescription of the b_{jj}, and are positive since $\Lambda = B^{-1}VB^{-1}$ is p.d.

Then (iii) follows from (16) and the definition of Π, while (iv) is an immediate consequence of (i), (ii).

3. We wish to find an upper bound to the probability P that the sample point x does not lie in the rectangular interval

$$|x_j| \leqslant \alpha_j \quad (j = 1, 2, \ldots, n), \tag{20}$$

which we shall denote by C. Let the equation

$$S(x) = 1, \tag{21}$$

where

$$S(x) = x'Ax = \sum \sum a_{jk} x_j x_k, \tag{22}$$

define an ellipsoid lying entirely inside C. The matrix A must thus be p.d., while, if the planes $x_j = \pm\alpha_j$ are not to cut the ellipsoid, we must have

$$a^{jj} \leqslant \alpha_j^2, \tag{23}$$

where the a^{jk} are the elements of A^{-1}.

Since all points outside C lie in the region $S(x) > 1$, we have, if $f(x)$ is the frequency function,

$$P \leqslant \int_{S>1} f(x)\,dx \leqslant \int_{S>1} S(x)f(x)\,dx \leqslant E[S(x)] = \sum \sum a_{jk} v_{jk}. \tag{24}$$

The margin of the inequality will be reduced if the ellipsoid fills C as

well as possible, i.e. if the planes (20) are tangent planes and equality holds in (23).

It is convenient to set $A^{-1} = B$ and summarize these results as follows:

THEOREM 1. *If the variates $x_1, x_2,..., x_n$ have zero means and covariance matrix V, and if $1-P = \text{prob}(|x_j| \leqslant \alpha_j; j = 1, 2,..., n)$, then*

$$P \leqslant \{\text{tr}(VB^{-1})\}, \tag{25}$$

where B is any p.d. matrix with diagonal elements given by

$$b_{jj} = \alpha_j^2. \tag{26}$$

The result could be extended to the case of asymmetric intervals $\beta_j \leqslant x_j \leqslant \alpha_j$; the ellipsoid would not then be central. If B is taken as purely diagonal, then (25) reduces to (6). In order to sharpen the inequality as far as possible, we now minimize with respect to B the expression on the right of (25), subject to the conditions imposed.

THEOREM 2. *The best possible inequality based on the first and second moments of a distribution with non-singular covariance matrix V is*

$$P \leqslant \{\text{tr}(V\bar{B}^{-1})\}, \tag{27}$$

where $\bar{B}$ is the unique solution of the equation

$$V = \bar{B}\Lambda\bar{B} \tag{28}$$

which fulfils the conditions of Theorem 1 (Λ being an unprescribed p.d. diagonal matrix). This inequality is equivalent to

$$P \leqslant \{\textstyle\sum \lambda_j \alpha_j^2\}. \tag{29}$$

By Lemma 3 expression on the right of (25) is minimum at its single stationary point. In order to locate the stationary point we set

$$0 = \frac{\partial}{\partial b_{jk}}\Big[\text{tr}(VB^{-1}) + \sum_l \lambda_l(b_{ll} - \alpha_l^2)\Big]$$

$$= -\text{tr}\Big(VB^{-1}\frac{\partial B}{\partial b_{jk}}B^{-1}\Big) + \lambda_j \delta_{jk} \quad (j, k = 1, 2,..., n), \tag{30}$$

the λ_j being Lagrangian multipliers. The passage from (30) to (28) is clear, and (29) follows from (28).

In order to prove that the inequality is best-possible, we must show that equality of P and $\{\text{tr}(V\bar{B}^{-1})\}$ can be attained.

Now, the coordinates of the point $x^{(j)}$ where the ellipsoid touches the plane $x_j = \alpha_j$ are

$$x_k^{(j)} = \bar{b}_{jk}/\alpha_j \quad (j, k = 1, 2,..., n). \tag{31}$$

If $\sum \lambda_j \alpha_j^2 = \rho^2 \leqslant 1$, consider the distribution which assigns probability mass $1-\rho^2$ to the origin and $\frac{1}{2}\lambda_j \alpha_j^2$ to the points $\pm(x^{(j)}+\epsilon)\,(j = 1, 2,..., n)$, ϵ being a vector with arbitrarily small positive elements. This distribution is readily found to have zero mean, covariance matrix $\bar{B}\Lambda\bar{B} = V$, and by construction fulfils the equality relation in (27). If, on the other hand, $\rho > 1$, consider the distribution which assigns probability mass $\lambda_j \alpha_j^2/2\rho^2$ to the points $\pm\rho x^{(j)}\,(j = 1, 2,..., n)$. For this distribution, which also fulfils the required conditions, $P = 1$, and the equality of (27) is again fulfilled.

4. The case $n = 2$ is easily treated explicitly since only the one constant $b_{12} = b_{21}$ ($= b$, say) is available for variation on the right of (25). Minimization leads to a quadratic equation in b whose smaller root must be chosen if B is to be p.d. Substitution of this root in (25) yields (7).

Inequality (7) has the following special cases:

$$v_{12} = 0: \qquad P \leqslant \left\{\frac{v_{11}}{\alpha_1^2}+\frac{v_{22}}{\alpha_2^2}\right\}, \tag{32}$$

$$v_{12}^2 = v_{11}v_{22}: \qquad P \leqslant \left\{\max\left(\frac{v_{11}}{\alpha_1^2}, \frac{v_{22}}{\alpha_2^2}\right)\right\}. \tag{33}$$

Case (33) is particularly interesting since the result is correct although V is singular. In §§ 2, 3 we excepted the case of singular V, which is not easily dealt with by the methods used there. Our results hold for V arbitrarily near to singularity, however, and, since it is intuitively obvious that the l.u.b. of P is a continuous function of V, the results yield valid bounds even for singular V.

In fact (7) is valid even in the exceptional case when $\mathrm{tr}(VB^{-1})$ has no stationary point at all, and the optimizing b is located on the boundary of Γ (rather, arbitrarily near the boundary). This happens when $v_{12}^2 = v_{11}v_{22}$ and $v_{11}/\alpha_1^2 = v_{22}/\alpha_2^2$. In this case

$$\Pi = \mathrm{tr}(VB^{-1}) = \frac{2|v_{12}|}{\alpha_1\alpha_2+b\,\mathrm{sgn}(v_{12})}. \tag{34}$$

This expression has no turning-point, but reaches its minimum value inside Γ against the boundary, for $b \to \alpha_1\alpha_2\,\mathrm{sgn}(v_{12})$ when

$$\Pi \to v_{11}/\alpha_1^2 = v_{22}/\alpha_2^2.$$

This limit is exactly the value given by equation (7).

5. I have not been able to solve the general equation (28) explicitly for $\bar{B}$, and it is unlikely that the general solution is simple enough to

be useful. However, if the bounds are not prescribed in advance, then V can be factorized in the form (28) in an infinite number of ways, and the diagonal elements of the resulting $\bar{B}$ will provide a set of numbers α_j. In any particular case it should be possible to find by experiment a factorization which yields bounds of approximately the right form for the purpose in hand.

The obvious factorization is

$$V = (c^2 V^{\frac{1}{2}})c^{-4}(c^2 V^{\frac{1}{2}}), \tag{35}$$

where c is a positive scalar and $V^{\frac{1}{2}}$ is the positive square root of V, i.e. the matrix obtained by taking the positive square root of all eigenvalues in the spectral representation of V. If the elements of $V^{\frac{1}{2}}$ are denoted by $v_{jk}^{(\frac{1}{2})}$, then we have

$$\alpha_j^2 = c^2 v_{jj}^{(\frac{1}{2})}, \tag{36}$$

$$P \leqslant \left\{\frac{\sum v_{jj}^{(\frac{1}{2})}}{c^2}\right\} = \left\{\frac{\operatorname{tr} V^{\frac{1}{2}}}{c^2}\right\}. \tag{37}$$

We obtain a more general factorization by taking any non-singular matrix H for which HH' is diagonal, and setting

$$\Lambda = HH', \tag{38}$$

$$\bar{B} = (H')^{-1}(H'VH)^{\frac{1}{2}}H^{-1}. \tag{39}$$

The factorization which is optimal, in that it yields the least P for a region C of given content $2^n \prod \alpha_j$, is readily found to be that for which $\lambda_j \alpha_j^2$ is constant. Of course, if one does not require C to be rectangular, then the optimal region is an ellipse $x'V^{-1}x =$ constant. The rectangular regions have a particular importance, however, since they correspond to fixed 'confidence bands' for the x_j.

6. As an example, let us consider a finite circulant process, i.e. assume that

$$v_{jk} = R_{j-k}, \tag{40}$$

$$R_j = R_{-j} = R_{n-j} \quad (j, k = 1, 2, \ldots, n). \tag{41}$$

If the *spectral ordinates* are defined by

$$F_j = \sum_{k=1}^{n} R_k e^{2\pi i jk/n}, \tag{42}$$

then the spectral representation of the covariance matrix V is

$$v_{jk} = n^{-1} \sum_{\nu=1}^{n} F_\nu e^{2\pi i (j-k)\nu/n}, \tag{43}$$

so that

$$v_{jj}^{(\frac{1}{2})} = n^{-1} \sum_{1}^{n} F_\nu^{\frac{1}{2}}. \tag{44}$$

The bounds α_j obtained by factorizing V as in (35) are thus independent of j. If P denotes the probability that any of the x_j fall outside the range $-\alpha \leqslant x \leqslant \alpha$, then we see from (37) and (44) that

$$P \leqslant \left\{\frac{1}{\alpha^2}\left[\sum_{\nu=1}^{n}\left(\frac{F_\nu}{n}\right)^{\frac{1}{2}}\right]^2\right\}. \tag{45}$$

The upper bound of P is thus determined by the 'square mean root' of the spectral ordinates. Equation (45) has the following special cases:

$$R_j = v\delta_{j0}: \quad P \leqslant \{nv/\alpha^2\},$$

$$R_j = v: \qquad P \leqslant \{v/\alpha^2\}. \tag{46}$$

Our proofs hold only for a finite set of variates, but it is interesting to examine the formal extension of the example to the case of a continuous circulant process, i.e. to a continuum of variates $x(t)$ $(0 \leqslant t \leqslant T)$ of zero mean for which

$$E[x(t_1)x(t_2)] = R(t_1-t_2), \tag{47}$$

$$R(s) = R(-s) = R(T-s) \quad (0 \leqslant s, t_1, t_2 \leqslant T). \tag{48}$$

In virtue of its periodicity and evenness $R(s)$ can be represented by a Fourier series

$$R(s) = \sum_{-\infty}^{\infty} \phi_\nu e^{2\pi i s\nu/T}, \tag{49}$$

where $\phi_\nu = \phi_{-\nu}$. Extract now the finite set of variates

$$x_j^{(n)} = x\left(\frac{jT}{n}\right) \quad (j = 1, 2, \ldots, n). \tag{50}$$

This constitutes a discrete circulant process with spectral ordinate

$$F_j^{(n)} = n \sum_{\nu=-\infty}^{\infty} \phi_{j+\nu n}. \tag{51}$$

By (45), the probability $P^{(n)}$ that any of $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ fall outside $-\alpha \leqslant x \leqslant \alpha$ satisfies

$$P^{(n)} \leqslant \{\Pi^{(n)}\} = \left\{\alpha^{-2}\left[\sum_{j=1}^{n}\left(\sum_{\nu} \phi_{j+\nu n}\right)^{\frac{1}{2}}\right]^2\right\}. \tag{52}$$

As n increases, $\Pi^{(n)}$ will converge to a finite limit if and only if the sum $\sum_{-\infty}^{\infty} \phi_\nu^{\frac{1}{2}}$ converges, in which case the limit is

$$\Pi^{(\infty)} = \alpha^{-2}\left[\sum_{-\infty}^{\infty} \phi_\nu^{\frac{1}{2}}\right]^2. \tag{53}$$

One is tempted to conjecture the more general result: that, if a stationary process $[x(t)]$ has spectral distribution function $F(w)$, then, provided that appropriate regularity conditions are fulfilled, a finite bound of

Tchebichev type can be set for the probability that $x(t)$ lies between prescribed finite bounds over a prescribed finite t interval if and only if the integral $\int [F'(w)]^{\frac{1}{2}}\, dw$ converges. (If $F'(w)$ is infinite for some w, then the integral must be suitably interpreted.) The conjecture is too academic to be worth following up, however, since it is possible (and in many cases necessary) to improve the Tchebichev bound considerably by the introduction and exploitation of relatively weak additional assumptions concerning the variate distribution function.

The construction of Tchebichev-type inequalities for random functions will be considered in a later publication.

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